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MODERN MATHEMATICAL SERIES
VOLUME II

General Editor

D. C. PAVATE, M.A. (Cantab.)

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GEOMETRY
AND
TRIGONOMETRY

for Pre-University Classes

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and

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Post & Telegraphs.

MACMILLAN AND COMPANY LIMITED
ADRAS BOMBAY CALCUTTA LONDON
1966

MACMILLAN AND COMPANY LIMITED
London Bombay Calcutta Madras Melbourne

THE MACMILLAN COMPANY OF CANADA LIMITED
Toronto

ST MARTIN'S PRESS INC
New York

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First Edition 1965
Reprinted 1966

MADE IN INDIA AT THE I.S.S.D. PRESS, MADRAS 2

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PREFACE

The content of mathematical teaching at the undergraduate stage is in urgent need of drastic change. The modern mathematics as developed during the last 50 to 100 years have hardly made any impression on the teaching of mathematics in schools and colleges. The main defect is that too much attention is paid to training in mathematical jugglery and too little to mathematical ideas. This has been realised by the University teachers of mathematics throughout India, who have been suggesting reforms in the curriculum and teaching methods from year to year. The University Grants Commission has also brought to the notice of the universities a revised curriculum in Mathematics and has recently issued a circular suggesting broad outlines of the reformed curriculum in mathematics for colleges including Pre-university classes. Since any attempt at modernization can proceed only step by step, the Karnatak University has adopted the suggested curriculum for Pre-university classes this year and hopes to change the first year course of the undergraduate stage next year and so on, from year to year, during the next three years.

The book contains an elementary course in Solid Geometry, a few theorems on similar triangles and proportional parts, an introduction to Analytical Geometry and the conventional Trigonometry course. When the syllabus was changed on the lines suggested by the University Grants Commission, the need for a suitable text-book in those subjects was felt. So I requested Professors C. N. Srinivasiengar and S. K. Singh of this University to write a suitable text-book for the use of students. The material provided by them has been partly re-written by me with a view to providing more attention to fundamental concepts and also to present the subject-matter in as simple a form as possible. The connection between Trigonometry and Analytical Geometry has been emphasized wherever possible. The present text-book is designed primarily to assist the students of mathematics in understanding the essential unity and simplicity of the subject.

Finally I hope that attempts will continue to be made by all Universities in India at modernizing their syllabuses so that the present hiatus between the standards in Mathematical education in India and other advanced nations will be bridged at the earliest possible moment.

Karnatak University,
Dharwar,
March, 1965.

D. C. PAVATE,
General Editor.

K. Subba Rao,
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PART I

CHAPTER 1

Lines and Angles

1. Introduction.

What is Geometry? Geometry like Arithmetic and Algebra started with material objects. When man first recognised form, shape and pattern of objects, Geometry came into being. The original meaning of Geometry was 'land measurement'. Actually Egyptians and Hindus used practical Geometry to a great extent as far back as 5000 years ago in measuring lands and constructing buildings. The land-surveyors were called rope stretchers, because they used a rope for measuring land and making a right angle. In ancient India, Geometry was called *Rajju*, which meant a rope. By means of such surveying methods, the Egyptians recognised certain relationships between the sides and angles of certain figures. The Greek scholars took interest in proving these relationships by logical deduction and in course of time developed Geometry as a logical science. Euclid wrote a series of books, about the year 300 B.C., called the *Elements*, in which he organised the then existing knowledge into a logical system. These books set the pattern for the study of Geometry for the next two thousand years. The Geometry you study in schools and colleges is, for the most part, fundamentally the same as the Geometry developed by Greeks and included by Euclid in his books.

Of course, Geometries other than Euclid's are possible. In 1637, René Descartes showed that methods of Algebra could be applied to Geometry. This is known as Coordinate Geometry. Under this system, points and lines have been denoted or even displaced by ' coordinates ' (x, y) or (x, y, z) or (l, m, n). Thus Geometry became a study of the algebra of ordered sets of numbers (x, y) or (x, y, z) etc. These numbers, too, need not be confined to the field of real numbers. They may belong to the field of complex numbers or to any other field. So we shall be dealing with ' imaginary ' points, lines etc. which have no meaning when applied to material objects. Similarly Geometry may be considered to be a branch of the theory of sets, by defining geometrical figures as sets of points or lines. The main thing about Geometry is that it should be based on logical reasoning. In the nineteenth century, many mathematicians (Gauss, Lobachevski and others) showed that non-Euclidean geometries could be developed logically. The German mathematician Hilbert published in 1899 the complete treatment of such geometries. Klein showed in 1872 that many of the Euclid's theorems could be applied, not only in ordinary space, but also to space-time. Later Einstein used this idea in his famous theory of Relativity. All this shows that Geometry has now become a very interesting and exciting subject, providing many opportunities of work and pleasure for thinking students.

2. Definitions and undefined terms.

In Geometry, as in every mathematical science, we come across concepts which are defined in terms of other concepts and these, in turn, are defined in terms of other concepts and so on; but we cannot go on with a ' chain of definitions '. We have to state precisely the undefinable *basic* concepts with the aid of which we can define other terms. For instance, the basic element of geometry is a *point*; it cannot be defined precisely. Usually we represent a point by a dot but we cannot see it physically because it has no *dimensions*. It is associated with the notion of *position* in *space*. Now space is visualized as the *set* of all points or positions. You see now how we use terms which cannot be precisely defined even

while describing one basic element viz. a point. We used the terms ' set ', ' dimensions ' which can only be described or vaguely imagined. There are many concepts in Geometry but we choose some as *basic*. We try to keep the number of basic concepts as small as possible. They remain undefined. The remaining concepts are defined with precision in terms of these basic concepts. They may be chosen in a number of ways, but usually we consider the following terms as *basic*:

Set, point, line, plane, dimension, between, belongs to, lies on, is equal to.

We are familiar with most of these terms in daily use. Some explanation is given below where necessary.

1. A set may be explained as a collection, or an aggregate, or a system or a family, but these are only alternative words which are equally undefinable. The pupils in a class form a set. All the players of a foot-ball match form a set, while the players of the two opposing teams form two sub-sets of this set. The spectators form another set and all the persons of the world who are not witnessing this particular match form another set. The umpire forms a set by himself, while the set of persons present in the field at midnight on the day of the match—there will be none—is called a null set. These are examples of finite sets. There are infinite sets too. For instance, the set of integers or the set of rational numbers. In Geometry, we shall be dealing with sets of points, lines, etc.

Space may be described as the set of all points or positions. A *plane* is thought of as a special subset of the points in space. Any set of points is called a figure.

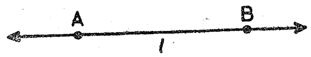

Fig. 1.1

A *line* is thought of as a special sub-set of points in a plane. It extends on both sides without limit. It is usual to name a line by the letter l or a particular line by l_1, and so on. Points A and B are said to be on the line l.

Definition : A line segment is the set of points lying between two given points A and B on the line.

This is an acceptable definition because it is expressed in words and phrases previously defined or which have been accepted as undefined. The terms 'points', 'line', 'set of points' and 'between' are all accepted as undefined here.

If the points A and B are included in the set of points of the segment AB of the line, the line segment AB will be called a *closed segment*. If they are excluded, the set of points lying between A and B will be called an *open* line *segment*. When we speak of a line segment, we usually include the end points.

A point is said to have zero dimension. Similarly a finite number of points A, B, C lying on the line l are said to form a set of zero dimension.

In particular, the end points A and B of a line segment are spoken of as a zero-dimensional segment. The segment AB consisting of all points between A and B with or without the end points, is called a *one-dimensional segment*. Imagine an ant crawling on a thin wire. Its position on the wire can be found in terms of the length travelled along the wire *i.e.*, its position depends on a single variable. The wire need not be in the shape of a straight line. It may be bent at several places as shown below.

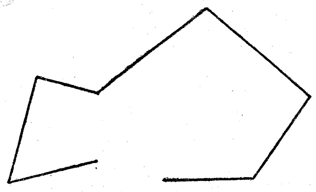

Fig. 1.2

Even then its length can be measured. Hence it is one-dimensional. If another ant can move across the top of a table, we can associate its position with two variables,—lengths along the two intersecting edges of the table. Hence the flat top of a table is *two-dimensional*. We can now imagine a thin plate instead of the top of a table. We may bend or twist this plate in any way. It will continue to be two-dimensional. To know the position of any point P on the plate, we need to know its distances from two perpendicular lines.

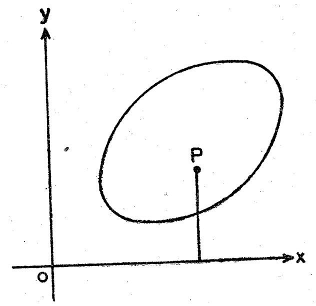

Fig. 1.3

Any two-dimensional object without thickness is called a *surface*. Similarly we can have three-dimensional objects such as a box which will be called solids. Consider a circle and a sphere. A set of points M whose distances from a given point O in the plane of this page are equal to a given segment of length r is a circle. It is a curve of one dimension. If we take the set of points whose distances from O are $\leqslant r$, the figure is a disc of two dimensions.

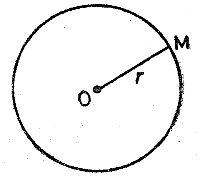

Fig. 1.4

A sphere is the set of all points which are at a given distance r from a given point O i.e. such that $OM=r$. A sphere is a surface. A solid sphere is the set of all points M which are at a distance such that $OM \leqslant r$. The figure is a globe. It is of three-dimensions. The two points P and P' at the end of a diameter of the sphere form a set of zero dimension.

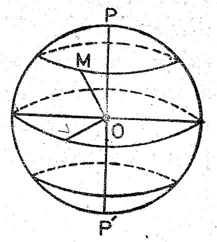

Fig. 1.5

Definition: *Ray.* A part of a line having one end point and extending without limit on one side of the end point is called a ray.

Fig. 1.6

Any point O on a line divides the line into two rays or two half-lines and the point O.

Definition of an angle: An angle is the set of all points on two distinct rays with a common end point. Here O is the common end point and OA and OB are the two rays. O is called the vertex. Each ray is called an arm of the angle. It is customary to name the angle as $\angle BOA$ or $B\hat{O}A$ (or $\angle AOB$ or $A\hat{O}B$).

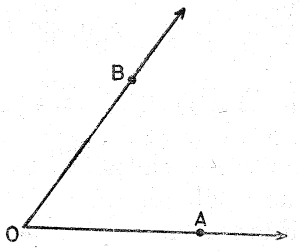

Fig. 1.7

Definition of a triangle: A triangle is the set of all points on the line segments determined by three points, which are not all on one line. Here A, B, C are three end points of the segments AB, BC and CA. A, B, C are called the vertices of the triangle and the line segments AB, BC and CA are called the sides of the triangle. If we extend the line segments indefinitely, they become rays and form three angles at A, B and C.

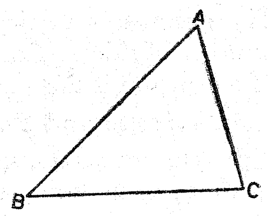

Fig. 1.8

Definition of a Polygon: A triangle is a particular case of a polygon. A figure like this is called a broken line figure. AB, BC, CD, DE, EF, FG, ... are line segments. When the beginning and end points coincide (in the figure A and G) and no two sides intersect, the line segments form a *polygon*. Thus a polygon may be defined as a closed broken line which bounds a portion of a plane.

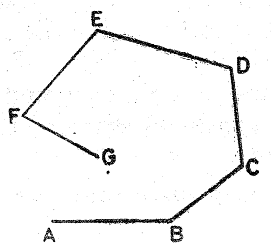

Fig. 1.9

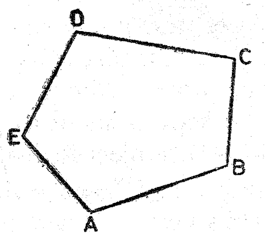

Fig. 1.10

1. A triangle is a polygon having three sides.
2. A quadrilateral is a polygon having four sides.
3. A pentagon is a polygon having five sides.
4. A hexagon is a polygon having six sides,
5. An octagon is a polygon having eight sides,
6. An n-gon is a polygon of n sides.

The number of vertices of a given polygon is the same as the number of sides. Consequently the number of internal angles of a polygon is the same as the number of sides.

3. Axioms and Postulates.

Just as we need undefinable basic concepts for the development of definitions, we need certain propositions which are accepted without proof. They may be called *assumptions* or *axioms* or *postulates*. The distinction between axioms and postulates is purely relative and not watertight. The axioms are accepted as self-evident truths which appeal to our intuition or commonsense. Postulates are truths which are not always so obvious to our intuition, but are truths to be accepted unquestioned for the development of the subject. In this book, we call all these assumptions as *postulates*.

In arriving at conclusions in geometry, we start with these postulates and prove the required theorems with the help of other theorems already proved, and the postulates we accept without proof. The number of these postulates should, however, be kept as small as possible, consistent with the need to develop the subject logically.

Postulate 1: **There exists one and only one line which contains two distinct points.**

Assuming this, we can prove that if two distinct lines intersect *i.e.*, have a common point, they do so in one point only. For, if there is another point, there will be two common points and by Postulate 1, there can be only one line lying on them. This contradicts what is given viz. the two lines are different.

A number of points are said to be collinear, if there is a line containing all of them.

4. The measurement of a line segment.

We usually measure a line segment by choosing the unit of length. The length of any line segment is found by counting the number of times the unit length is contained in it. In theoretical geometry, however, a line segment is a set of points which cannot be physically measured. To associate a length with a line segment, we need the following postulate:

Postulate 2: **There is a one-to-one correspondence between the points on a line and the real numbers. That is, to each point on l, there corresponds a single real number and conversely.**

This postulate enables us to associate a length with a line segment. We choose any arbitrary point O to represent the number zero and another suitable point A to represent 1. Now the segment OA represents the unit of length. We associate points with numbers 2, 3, 4, ... on the right side of OA and with numbers -1, -2, -3, ... on the left side of OA. These real numbers are referred to as the *coordinates* of the points on the line. Suppose B has a coordinate 5 and C has a coordinate -2. The length of BC is 7 units, for $5-(-2)=7$.

5. Betweenness:

When we say that a point C is *between* A and B, we mean

(i) all three points lie on the same line, and
(ii) $AC+CB=AB$.

From this and properties of real numbers, it follows that if C is between A and B, $AB=AC+CB$ and $AC=AB-CB$.

Postulate 3: **If three distinct points lie on a line, one and only one of the points is between the other two.**

From Postulates 2 and 3, it follows that there is one and only one point C between A and B such that $AC=CB$. C is the midpoint of the line segment AB. This shows that there is only one point on the line which represents $\frac{1}{2}$ or $\frac{3}{2}$ or $\frac{1}{4}$ or any fraction. Similarly it can be shown that there is one distinct point which divides AB at a point C in a given ratio. Thus the points on the ray OA may be placed in one-to-one correspondence with the positive real numbers on the right side of OA and the negative real numbers on its left side. If we take two points A and B on l corresponding to the real numbers a and b, the set of all points 'x' on l corresponding to the numbers $a \leqslant x \leqslant b$ is the line segment AB on l. If the points A, C, B on l correspond to the real numbers a, c, b and if $a < c < b$, then C is said to be between A and B or to be an internal point of the segment AB. In this case we may speak of B being *external* to AC.

6. The measurement of an angle.

An angle was defined as the set of points of two distinct rays and the common end point. As in the case of the line segment, angles, thus defined, cannot be physically measured. However, we associate the idea of size with an angle. The measurement of an angle should describe the relative position of its rays. So we have to choose a unit angle. The unit angle is based on the straight rays, OA and OB on opposite directions. The straight angle is divided into 180 units called degrees. One degree $=\frac{1}{180}$ of a straight angle. To measure any angle we use the protractor on which all the 180 degrees are marked. We accept intuitively the following Postulate:

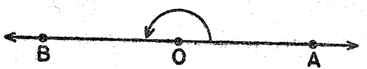

Fig. 1.11

Postulate 4: **To each geometric angle there corresponds a unique positive real number of degrees; the degree measure of an angle is greater than zero and equal to or less than 180.**

For purposes of geometry, the idea of an angle as a point set, with its measure from 0° to 180° is sufficient. Later we shall define an angle in other ways.

Since the measurement of an angle is a positive real number in degrees, angles may be added and subtracted according to the properties of real numbers. For instance, if there are any two angles, $\angle AOB=50°$ and $\angle BOC=60°$, then $\angle AOC=\angle AOB+\angle BOC=110°$. The ray OB in the figure is between ray OA and ray OC; for, B is on the same side of OA as C and B is on the same side of line OC as A. B is said to be in the interior of $\angle COA$. In this case

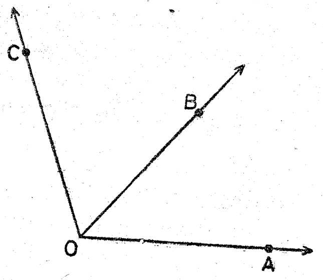

Fig. 1.12

(i) $\angle COA=\angle COB+\angle BOA$

(ii) $\angle COB=\angle COA-\angle BOA$

(iii) $\angle BOA=\angle COA-\angle COB$

These results follow from the postulate on betweenness.

Just as there is only one midpoint of a line segment, there is only one ray bisecting a given angle. In the figure above, if $\angle BOA=\angle COB$, the ray OB bisects the angle COA. Similarly there is only one ray OB which bisects a straight angle COA. That ray is said to be **perpendicular** to the line AC. Each of the two angles viz. angle COB and angle BOA is said to be a **right angle.**

Since a unique number 90 is associated with a right angle, all right angles are equal. Two rays, AB, AC which contain a right angle, are said to be perpendicular to each other.

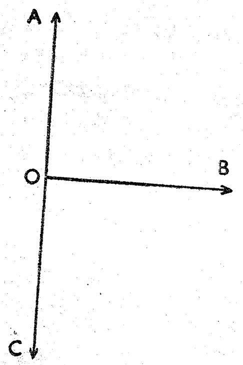

Fig. 1.13

Definition : Two angles whose angle sum is 90° are complementary angles. Two angles whose angle sum is 180° are called supplementary angles. The statement *AB* is perpendicular to the line *CD* is expressed by the symbol $AB \perp CD$.

7. Parallel Lines.

Definition : If two distinct lines lie in the same plane and have no point in common, then the lines are said to be parallel to each other. The symbol || stands for 'is parallel to'. The line segment *AB* is said to be parallel to the line segment *CD*, if *AB* and *CD* lie in the same plane and do not intersect, however far they are produced in either direction.

Parallel postulate: **If *l* is any line and *A* is any point not on *l*, then there is one and only one line through *A* which is parallel to the line *l*.**

This is often referred to as 'Euclid's fifth postulate' but actually Euclid's postulate is in a slightly different form.

Euclid's postulate. 'Let *l* be an arbitrary line, and *A* a point not on *l*; then in the plane determined by the line *l* and the point *A*, there does not exist more than one line passing through the point *A* and not intersecting the line *l*.'

The postulate does not state whether even one such line exists; but we can prove that there exists, in the plane determined by the line *l* and the point *A*, at least one line passing through *A* and not intersecting *l*.

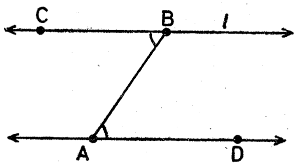

Fig. 1.14

Take any points *B* and *C* on the line *l* and join *BA*. Through *A* draw a line on the plane so that $\angle CBA = \angle BAD$.

This line *AD* does not intersect *l*, however far the lines *AD* and *CB* are produced. If they do at *P*, the figure will be as

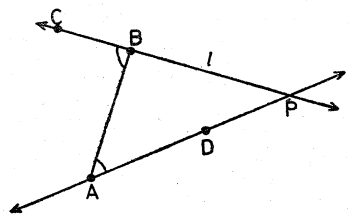

Fig. 1.15

shown here. In the triangle *ABP*, the exterior angle *ABC* = the interior angle *BAP*. This is not possible.

∴ our assumption that the line *AD* meets *l* is false.

∴ there is one line passing through *A* which does not meet *l*.

Combining this result with Euclid's postulate, we can state that there exists one and only one line through *A* in the plane which does not intersect the line.

This incidentally shows that the alternate angles made by a transversal (a straight line like *AB*) cutting two parallel lines are equal.

In this figure angles 1 and 3, 2 and 4, 5 and 7, 6 and 8 are called corresponding angles. Angles 2 and 7, 6 and 3 are called alternate angles. It follows that corresponding angles are

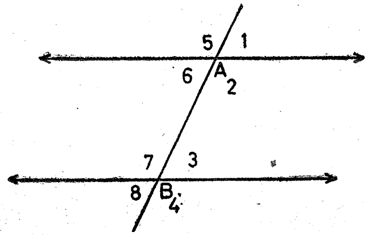

Fig. 1.16

also equal; for $\angle 6 = \angle 1$ (vertically opposite angles) and if $\angle 6 = \angle 3$, $\angle 1 = \angle 3$.

Some people prefer to define parallel lines as straight lines in the same plane that form equal corresponding angles with a third line.

The parallel postulate was expressed by a British mathematician John Playfair in 1858, in *yet another form.* Two intersecting straight lines cannot both be parallel to the same straight line. This follows from the postulate in the form given above viz. if l is any line and A is any point not on l, there exists one and only one line through A which is parallel to the line l. Two lines through the point of intersection cannot be parallel to l.

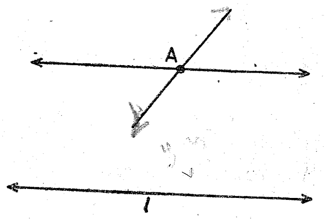

Fig. 1.17

Calculation of the circumference of the earth. The Parallel postulate—Euclid's Fifth postulate—was used by Eratosthenes (about 240 B.C.) to calculate the circumference of the Earth. On a certain day at noon, the sun shone just overhead at Aswan on the Nile, while at the same time, at Alexandria 800 km south of Aswan, the shadow of a pillar showed that the sun was 7° 12′ south of the vertical.

At O, the centre of the earth, the arc AS makes $7^\circ\,12'$ or $\frac{36}{5}$ degrees as seen from the figure. The circumference of the earth is $\frac{360}{\frac{36}{5}}$ or 50 times AS.

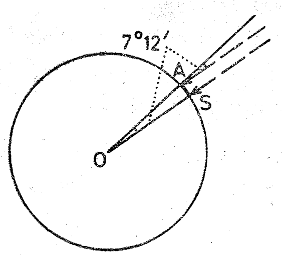

Fig. 1.18

$$\therefore \text{circumference} = 50 \times 800 \text{ km.}$$
$$= 40000 \text{ km.}$$

This is almost a correct result, calculated more than 2000 years ago. Here the geometrical fact used was $\angle AOS$ = the difference in the directions of the sun's rays from A to S, as they are corresponding angles.

8. Inductive and Deductive reasoning.

A student engaged in scientific research makes careful observations in the laboratory on the basis of many experiments and arrives at some generalisation or conclusion. For instance, one can infer, by measuring and finding the sum of the three angles of different triangles that the angle sum of a triangle is two right angles or 180°. This process is called inductive reasoning. Although many discoveries are the result of inductive reasoning, some of the conclusions reached may be wrong. Inductive reasoning proceeds from particular cases to a generalisation. The conclusions may not be always valid. On the other hand, in Mathematics, we always expect conclusions through **reasoning.** Consider the statements.

Premises { 1. If Mathematics is well taught in the college, Ramesh will master it.
2. Mathematics is well taught in this college.

Conclusion 3. Ramesh will master Mathematics.

A formal deductive proof is a succession of statements which lead to the required conclusion. In Geometry, the definitions, postulates or axioms and the results previously proved are the basic implications from which other geometric relationships are deduced by logical reasoning.

Exercise 1

1. Name the set to which each of the following belongs:
(a) Delhi, (b) plane Geometry, (c) a rose,
(d) a triangle, (e) ray, (f) angle.

2. Name the postulates required to prove the following:

Line 1 contains points A, B.
Line 2 contains point A.
Line 2 does not contain B.

3. One of the assumptions is that a straight line segment is the shortest line segment that can be drawn between two points. With this assumption, prove that $AB+BC > AC$ in a triangle.

4. A real number coordinate system is a set up on a line. What is the distance between the pairs of points with the coordinates as shown below.

(i) 2, 6; (ii) -2, 3; (iii) 0, -2; (iv) -3, -5, (v) -5, -3.

5. The coordinates of two points A and B on a line are 3, 7 according to our coordinate system.

What would be their coordinates if the coordinate system were shifted two units to the right? two units to the left? Would the distance between A and B change if the coordinate system were changed?

6. How many rays bisect an angle? How many rays bisect a straight angle? What is the measurement of each of the angles in the latter case?

7. Show by deductive reasoning that the supplements of equal angles are equal.

8. Show by deductive reasoning that the complements of equal angles are equal.

9. Find the number of degrees in an angle which is three times its complement.

10. Prove that $\angle x = \angle y$ in this figure, if $\angle u = \angle v$. Here $OX \perp OY$.

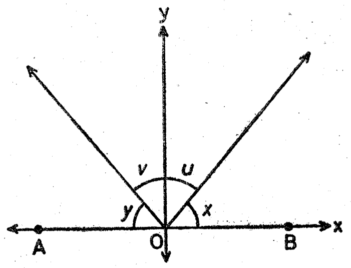

Fig. 1.19

11. Defining parallel lines as straight lines in the same plane that form equal corresponding angles with a third line, prove by deductive reasoning that two parallel lines have no common point.

12. If the bisector of an exterior angle of a triangle is parallel to a side, prove by deductive reasoning that the triangle is isosceles *i.e.*, its two angles are equal.

CHAPTER 2

Planes

A plane is described by means of the following postulates.

1. A plane is a two-dimensional set of points and contains at least three non-collinear points.

2. Any three non-collinear points determine a plane.

3. If two distinct points *P* and *Q* lie in a plane, then the line containing these points lies in the same plane.

4. An unlimited number of lines in one plane may lie on one point in the plane.

5. If two distinct planes intersect, they intersect in a line.

More generally, if two distinct surfaces intersect, they intersect along a *curve*. This curve is called a section of one surface by the other. Although a plane is unlimited in extent in space, we usually show it on a piece of paper as a quadrilateral, more often as a parallelogram.

We refer to a plane as plane *ABCD* or simply as plane *AC* or even more simply as plane *p*.

Fundamental properties of planes.

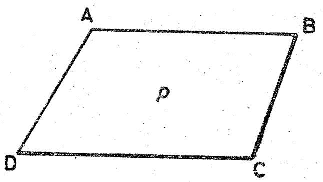

Fig. 2.1

Theorem 1

A line and a point not on it determine one and only one plane.

Three points which are non-collinear determine a plane. If we take any two points *B*, *C* on the line *l*, the three points *A*, *B*, *C* determine a plane. The line *l* and the point lie on this plane.

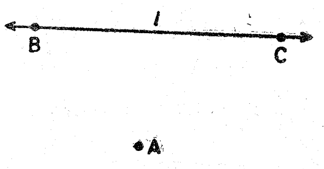

Fig. 2.1a

Theorem 2

If two lines intersect, they lie in a definite plane.

Three points determine a plane. We can take the point of intersection and one other point on each of the two lines and fix the plane.

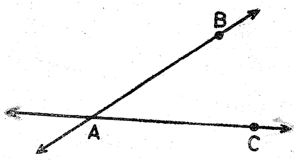

Fig. 2.2

Theorem 3

If a line intersects a plane not containing it, it intersects in a single point.

If the line intersects the plane in more than one point, it must wholly lie on the plane according to Postulate 3 above. Since it is given that the plane does not contain the given line, it may intersect at the most in a single point.

It is possible that a line may not intersect the plane at all. In that case it is said to be parallel to the plane.

As we have seen earlier, by definition, parallel lines are in the same plane and do not intersect. It is possible to have two non-parallel straight lines not in a plane which do not intersect. They are *skew* lines.

Definition: Two lines which are not coplanar are said to be *skew* lines. For instance, one leg of a table and any line on the top of the table not meeting this leg, form skew lines.

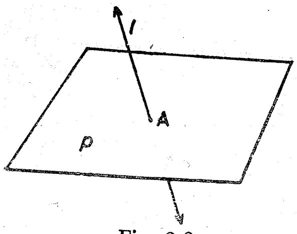

Fig. 2.3

Here the line l intersects the plane p at the point A.

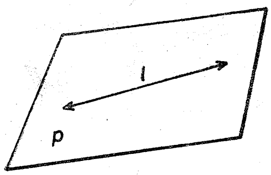

Fig. 2.4

Here the line l wholly lies in the plane p.

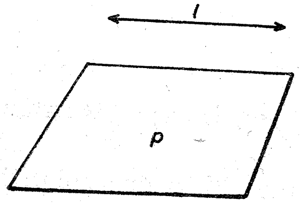

Fig. 2.5

In this figure the line l does not intersect the plane at all. It is said to be parallel to the plane.

From Postulate 5 above, two planes may intersect in a line. When they do, they form an angle. Such an angle is called *a dihedral angle* (Fig. 2.6). The two planes AD and BC, are called *faces* of the dihedral angle. The line of intersection l is called the *edge*. The measure of a dihedral angle is obtained as follows. At any point O on the edge, draw OP, OQ perpendicular to the edge in the two planes respectively. Then POQ is the angle between the planes or the dihedral angle.

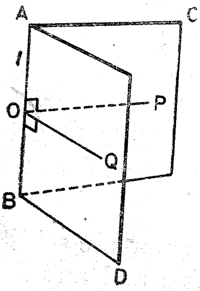

Fig. 2.6

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Theorem 4

If a line l in a given plane is parallel to a line m not on the plane, then m is parallel to the plane.

Proof: The lines l and m are coplanar, $\because$ they are parallel. The plane containing them meets the given plane along l. Let m meet the given plane, if possible, at a point P. Then P lies on both the planes, and hence on l. Thus l and m meet at a point P, contrary to the hypothesis that l and m are parallel. Hence m cannot meet the given plane in a point. $\therefore$ the line m is parallel to the plane.

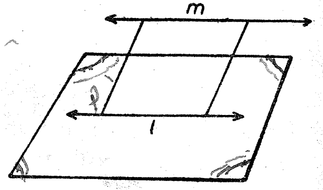

Fig. 2.7

Theorem 5

If m is a line parallel to a given plane, any plane through m meets the given plane along a line l which is parallel to m.

If l and m were to meet in P, then m would meet the given plane at P, contrary to hypothesis.

8. *Postulate* 6. **If O is a point on a plane, there exists one and only one line through O which is perpendicular to all lines in the plane which pass through O.**

This line is called the *normal* at O. If P is any point on this, PO is the *normal to the plane from P*.

Theorem 6

If O is a point on a plane, and OP a line such that it is perpendicular to two lines through O in the plane, then OP is the normal at O to the plane, i.e., OP is perpendicular to all other lines through O in the plane.

Given: OA, OB are two lines in a plane, and $OP \perp$ both of them.

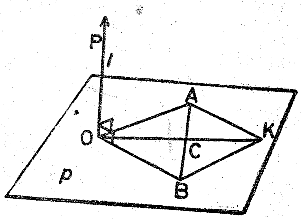

Fig. 2.8

Let OK be any other line through O in the plane.

To prove that: $OP \perp OK$.

Proof: Take any points A, B, K on OA, OB, OK so as to complete the parallelogram $OAKB$. Let OK and AB meet at C.

$\therefore AC=CB$.

From $\triangle PAB$, $PA^2+PB^2=2AC^2+2PC^2$ (Theorem of Apollonius)

From $\triangle OAB$, $OA^2+OB^2=2AC^2+2OC^2$.

Subtracting,

$$(PA^2-OA^2)+(PB^2-OB^2)=2(PC^2-OC^2)$$

Since POA, POB are right angles,

$PA^2=OP^2+OA^2$, and $PB^2=OP^2+OB^2$ (Pythagoras' Thm.)

$\therefore 2\,OP^2=2(PC^2-OC^2)$

$\therefore OP^2+OC^2=PC^2$

$\therefore OP \perp OC$ *i.e.*, to OK.

Theorem 7 (Converse of Theorem 6)

All lines perpendicular to a given line through a given point on it are coplanar.

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Given: $OP \perp OA, OB, OC$.

To prove that: OA, OB, OC are in the ame plane.

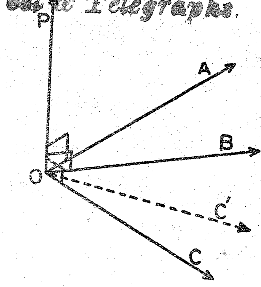

Fig. 2.9

Proof: If possible, let the plane AOB meet the plane POC along the line OC', different from OC.

By Theorem 6, $OP \perp OC'$.

Hence OC and OC' are two lines $\perp$ OP *in the same plane* POC. This is impossible.

$\therefore OC'$ coincides with OC

i.e., OC lies in the plane AOB

Similarly any other line $\perp OP$ at O lies in the plane AOB.

Theorem 8 (a)

Any two normals to a given plane are parallel.

Given: OP and $O'P'$ are normal at O and O' respectively to a given plane.

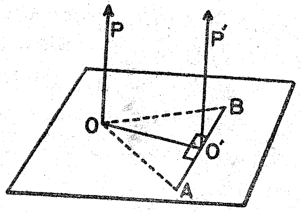

Fig. 2.10

To prove that: $OP \parallel O'P'$.

Construction: In the given plane draw the line through $O' \perp OO'$, and cut off equal lengths $O'A$, $O'B$ on opposite sides.

Proof: $\triangle$s $OO'A$ and $OO'B$ are congruent, since $O'A=O'B$, and OO' is common and $\perp AB$.

$\therefore OA=OB$

$\therefore \triangle$s POA, POB are congruent; since $OA=OB$, PO is common, and $P\hat{O}A=P\hat{O}B=90°$, for PO is normal to the plane.

$\therefore PA=PB$.

$\therefore \triangle$s $PO'A$, $PO'B$ are congruent.

$\therefore PO' \perp AB$.

Hence $AB \perp O'O, O'P, O'P'$

$\therefore$ the plane POO' contains the line $O'P'$.

In this plane, $P\hat{O}O' = O\hat{O'}P = 90°$.

$\therefore OP \parallel O'P'$

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Theorem 8 (b) (Converse Theorem)

If $OP \parallel O'P'$, and OP is normal to a plane, then $O'P'$ also is normal to the plane.

The proof of Theorem 8(*A*) has to be changed only in the last few lines. We prove $PO' \perp AB$, as before. Now OP, $O'P'$ are coplanar, and $AB \perp OO'$ and $O'P$.

$\therefore AB$ is normal to the plane $POO'P'$ at O.

$\therefore O'P' \perp AB$ as also to $O'O$.

$\therefore O'P'$ is the normal to the plane at O'.

Corollaries:

Cor. 1: We have proved above that $PO' \perp AB$, given PO normal to the plane and $OO' \perp$ a line AB in the plane. We enunciate this as follows:

If from any point P outside a plane, PO is drawn normal to the plane, and if OO′ is drawn perpendicular to a line AB in the plane, then PO′ is perpendicular to AB.

Cor. 2: **If from any point P outside a plane, PO is drawn normal to the plane, and if PO′ is drawn perpendicular to a line AB in the plane, then OO′ is perpendicular to AB.**

The student will easily make the needful changes in the proof above.

Corollaries 1 and 2 are two different forms of *The Theorem of Three Perpendiculars*.

A slight variation of this Theorem gives the following method of practical construction for drawing the normal to a plane from any external point.

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Let P be any point outside the plane p, and l any ine in the plane. Draw PO′ ⊥ l, and draw the line hrough O′ in the plane, perpendicular to l. If PO is lrawn perpendicular to this line, then PO is the normal from P to the plane.

Proof: Let B be any point on l. Referring to Fig. 2.10 we have

$$\begin{aligned} OB^2 &= OO'^2 + O'B^2 \\ &= OO'^2 + (PB^2 - PO'^2) \\ &= PB^2 - (PO'^2 - OO'^2) \\ &= PB^2 - PO^2. \end{aligned}$$

$\therefore OB^2 + OP^2 = PB^2$

$\therefore PO$ is $\perp$ to OB

Thus PO is $\perp$ to OO' and OB.

Hence PO is the normal at O to the plane.

Exercise. Prove the following construction for drawing the normal to a plane at a point O on it: Draw $OO' \perp$ to some line l in the plane, not passing through O. At O', draw any line $O'Q \perp l$, outside the plane. From any point Q on this, draw $QR \perp$ to OO'. Then the line through O parallel to RQ is the normal at O to the plane.

Theorem 9

If a line l is parallel to a plane, the normal distances of any two points on l from the the plane are equal.

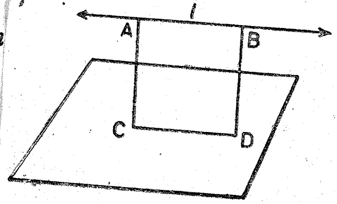


Fig. 2.11

Given: A, B are two points on l, and AC, BD the normals from them to a plane parallel to l.

To prove that: $AC = BD$.

Proof: $AC \parallel BD$, by Theorem 8(A)

$\therefore A, C, D, B$ lie in one plane.

AB must be parallel to *CD*, for if *AB* meets *CD* in a point, then *AB* will not be parallel to the plane. Hence *ACDB* is a rectangle, since the opposite sides are parallel, and $A\hat{C}D=90°$

$\therefore AC=BD$.

Theorem 10

If the line l || m, and m || to n, then the lines l and n are parallel.

Proof: Draw any plane normal to *l*. Let this cut the lines *l*, *m*, *n* in *A*, *B*, *C* respectively.

Then *m* is normal to the plane at *B*. (Theorem 8(b))
and *n* is normal to the plane at *C* (")
$\therefore$ *l* and *n* are parallel (Theorem 8(a)).

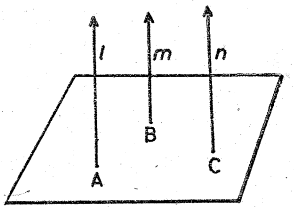


Fig. 2.12

9. Angle between two skew lines.

Let *AB*, *CD* be two skew lines. Through any point *A* in *AB* draw *Ad* parallel to *CD* (see Euclid's Postulate). Then the angle $B\hat{A}d$ is defined as the angle between *AB* and *CD*.

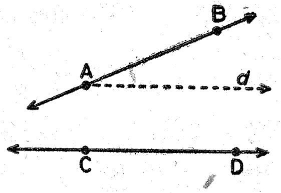


Fig. 2.13

10. *Orthogonal Projection.* If from any point *P*, the line *PP′* is drawn normal to a given plane, the foot *P′* of this normal is called the *orthogonal projection* (or simply the *projection*) of *P* on the plane. If *P* moves on a curve, the projection *P′* will trace another curve which is called the *projection of the given curve* on the plane.

Theorem 11

The projection of a straight line on a plane is a straight line.

A, *B* are any two points on the given line *l*, and *A′*, *B′* their projections.

AA′, *BB′* are therefore normals to the plane and hence are parallel and lie in one plane. If *A* be taken as fixed on *l*, the plane *AA′B′B* is the fixed plane through *AA′* and *l*. The projection of any point *B* on *l* lies on the line of intersection of this plane and the given plane. The segment *A′B′* is said to be the projection of the segment *AB*.

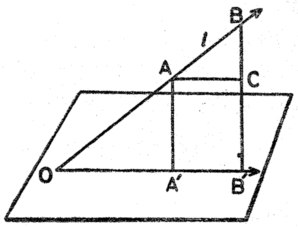


Fig. 2.14

Cor. 1: If *AB* is parallel to the given plane, its projection *A′B′* is parallel to *AB*, and *A′B′* = *AB*.

For *AA′B′B* is now a rectangle.

Cor. 2: If two lines are parallel, their projections also are parallel (or coincident). For if the projections meet in a point, this point is the projection of a point on each of the given lines, *i.e.*, the given lines intersect, contrary to hypothesis.

11. We are now in a position to define the angle between a line and a plane.

The angle between a line and a plane is defined as the angle between the line and its orthogonal projection on the plane.

Thus, in Fig. 2.14, if *AB* meets the plane in *O*, the projection of *O* is *O* itself, and hence *O*, *A′*, *B′* are collinear. The angle between *AB* and the given plane is the angle *A′OA*. It is easily seen that this angle is the complement of the angle that *AB* makes with the normal to the plane. If $A'\hat{O}A=\theta$, it is easily proved that

$$A'B' = AB \cos \theta,$$

for draw $AC \parallel A'B'$ to meet BB' in C. Then $AA'B'C$ is a rectangle.

$$\therefore A'B' = AC = AB \cos BAC = AB \cos \theta.$$

Hence,

the length of projection of a segment of a line = length of the given segment multiplied by the cosine of the angle between the given line and the plane.

12. The common perpendicular to two skew lines.

Gup

Theorem 12

One and only one line can be drawn so as to intersect two skew lines and be perpendicular to both of them. The line is the shortest distance between the skew lines.

Construction: l and m are two skew lines. Through any point H on m, draw the line $n \parallel l$. Then l is parallel to the plane formed by m and n (Theorem 4). Let l' be the orthogonal projection of l on this plane. Let l' cut m in M. Let L be the point on l whose projection is M.

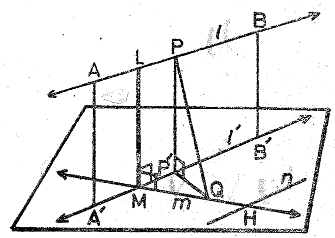


Fig. 2.15

Then LM is the required line.

Proof: By construction, $LM \perp m$.

Let A, B be two points on l and A', B' their projections.

$\therefore AA' \parallel BB'$, and $AB \parallel A'B'$ (Theorem 11, cor. 1)

$\therefore AA'B'B$ is a rectangle.

Similarly $AA'ML$ is a rectangle.

$\therefore ML \perp AB$.

Thus, $LM \perp$ both l and m.

Next, if PQ be any other line joining l and m, and if P' is the projection of P, $PP'Q$ is a triangle, right-angled at P', and hence $PQ > PP'$

But $PP' = LM$ $\quad \therefore PQ > LM$.

Thus LM is the shortest line joining l and m.

13. We collect together, in this section, a number of important results, in the form of problems, outlining the method of proof.

(1) **If two parallel planes are cut by a third plane, the lines of intersection are parallel.**

Use Reductio ad absurdum method.

(2) **If a line is normal to two different planes, the planes are parallel.**

If possible let C be a point common to the planes. Let the line meet the planes in A and B. Consider $\triangle ABC$.

(3) **Any plane through a normal to a plane is perpendicular to the plane.**

Let OC be the normal, and OA the line of intersection of the two planes. Let OB be the normal to plane COA. Angle between the two planes, by definition $= C\hat{O}B = 90°$.

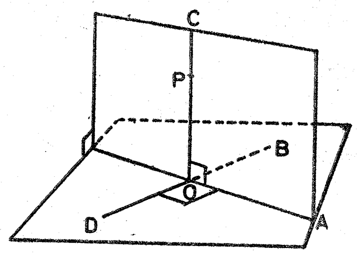


Fig. 2.16

(4) **If two intersecting planes are each perpendicular to a third plane, their line of intersection is normal to the third.**

We may use Fig. 2.16. Suppose AOC, BOC are two planes both $\perp$ the third plane AOB. [But AOC, BOC need not be $\perp$ each other.]

In the plane AOC, draw $OP \perp OA$, and in the plane AOB, draw $OD \perp OA$. $\therefore P\hat{O}D=90°$.

$\therefore PO$ which $\perp AO$ and OD is the normal at O, to the plane AOB. Thus the normal at O to the plane AOB lies in the plane AOC. In a similar manner it lies in the plane BOC. Hence the normal coincides with OC.

(5) **If two intersecting lines in one plane are respectively parallel to two intersecting lines in another plane, the angles between the two pairs of lines are equal.**

Given that l, m are two lines meeting at O. l', m' are two lines in another plane parallel to these respectively and meeting at O'. Let the normal at O' to the plane $l'm'$ meet the plane lm at A. Draw l'', m'' through $A \parallel l$, m respectively.

Prove that $O'A \perp l'$, m', l'', m''. Hence these four lines are coplanar. $\therefore \hat{O'}=\hat{A}$. But $\hat{A}=\hat{O}$.

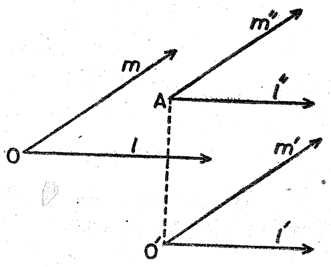


Fig. 2.17

(6) **The angle between a line and a plane is the least angle made by the line with any line in the plane.**

Let the line AB meet the plane at O and $A'B'$ the projection of AB. By definition, angle between AB and the plane is $A'\hat{O}A$.

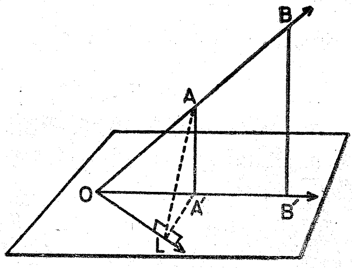


Fig. 2.18

Let OL be any other line in the plane. [If it does not pass through O, we take a line through O parallel to it. See Definition of angle between skew lines.]

Draw $A'L \perp OL$. Then $AL \perp OL$ (Theorem of 3 perpendiculars.)

$$\text{Now, } \cos A'\hat{O}A = \frac{OA'}{OA}, \quad \cos LOA = \frac{OL}{OA}.$$

But, from the rt. ∠d $\triangle OA'L$, $OA' > OL$

$\therefore \cos A'OA > \cos LOA$

$\therefore A'OA < LOA$.

(7) **The line of greatest slope on a given inclined plane is the line perpendicular to the line in which the plane meets the horizontal.**

LM is the line of intersection of an inclined plane and the horizontal plane. From any point A on the inclined plane draw $AL \perp$ to LM. Let AA' be the normal to the horizontal. Then

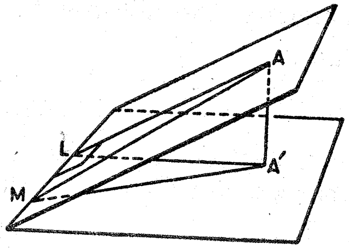


Fig. 2.19

$$\sin ALA' = \frac{AA'}{AL}, \quad \sin AMA' = \frac{AA'}{AM}$$

But $AM > AL$, since $ALM=90°$

$\therefore A\hat{L}A' > A\hat{M}A'$.

(8) **Given two skew lines, one and only one line can be drawn through any point O outside these lines, so as to intersect both of them.**

Consider the planes through O and each of the given lines. Their line of intersection is the required line.

(9) We have already defined the dihedral angle or the angle between two planes. **The angle between two planes can also be defined as the angle between their normals.**

Prove the equivalence of the two definitions, as follows.

From any point O not on either plane, draw the normals OP, OQ to the planes. Draw $PL \perp$ the common line AB. Then LO and $LQ \perp AB$. Hence LP, LO, LQ are in one plane. Hence $P\hat{L}Q$=supplement of $P\hat{O}Q$. ($\because$ $PLQO$ is a cyclic quadr.) *i.e.*, one of the angles between the normals OP and OQ to the two planes is equal to the dihedral angle $P\hat{L}Q$.

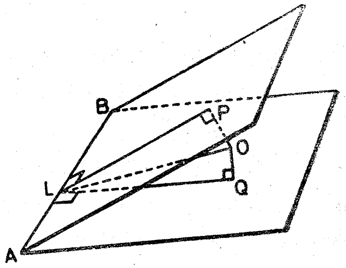


Fig. 2.20

Exercise 1

1. Explain why a stool having three legs is always steady, while a chair with four legs may not be steady.

2. Prove that through a given external point, only one plane can be parallel to a given plane.

3. Show that intersections of two parallel planes with a third plane are parallel lines.

4. A line parallel to each of two intersecting planes is parallel to their line of intersection.

5. If two lines are parallel, a plane which contains one of the lines and only one, is parallel to the other line.

6. Prove that lines drawn to a plane from any point P on the normal to the plane so as to cut off equal distances in the plane from the foot of the normal are equal.

7. When a line and a plane are parallel, another line parallel to the first, through a point in the plane, lies entirely in the plane.

8. A line that is perpendicular to one of two parallel planes is perpendicular to the other.

9. In the diagram, Fig. 2.21, $AO \perp$ plane p, $OB \perp OC$,

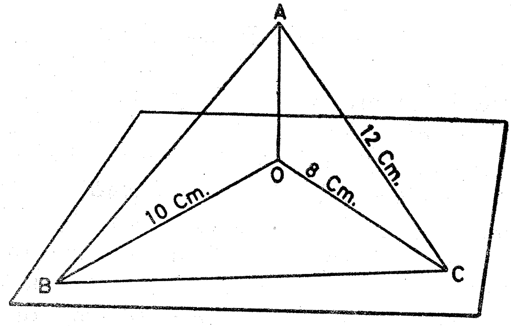


Fig. 2.21

AC=12 cm, OC=8 cm, and OB=10 cm. Calculate the length of AB.

10. If AB, CD are skew lines, AC, BD and AD, BC are also skew.

11. One and only one plane can be drawn through a given point so as to be parallel to each of two skew lines.

12. There are two parallel planes each containing one of two given skew lines.

13. The middle points of the sides of a skew quadrilateral form a parallelogram whose plane is parallel to both the diagonals of the quadrilateral.

14. Prove that the measure of a dihedral angle (see p. 28) is independent of the choice of the point O.

CHAPTER 3

Ratio and Proportion

1. We often compare two or three quantities of the same kind by means of ratios. The ratio of two like quantities is the quotient of their measures in terms of the same unit. If the ages of two boys Manohar and Shreedhar are 12 and 15 years respectively, the ratio of their ages is $12 : 15$ or $\frac{12}{15}$ or $\frac{4}{5}$. This is often written in the form: the age of Manohar: the age of Shreedhar$=12 : 15$ or $4 : 5$. We can cancel the common factors of a ratio, as we do in the case of a quotient; for, the ratio is another name for a quotient. Similarly if the ages of three boys A, B, C are 12, 15, 18, the ratio of their ages is written : the age of A : the age of B : the age of $C=12 : 15 : 18$ or $4 : 5 : 6$. In general $x : y : z=a : b : c$ if and only if there is a real number k, which is not zero, such that

$$x=ka, \quad y=kb, \quad z=kc.$$

A proportion is a statement that two ratios are equal. Thus if $x : y=a : b$, then x, y, a, b are said to be in proportion. The statement $x : y : z=a : b : c$ is also called a proportion.

In Geometry, we compare lengths of line segments or areas of two different figures and so on.

2. Division of a line segment in a given ratio.

You are already familiar with the notion of the length of a line segment in suitable units such as cm, inch, etc. We also set up a one-to-one correspondence between real numbers and points on a line. We have also seen that a point C is between two points A and B, if $a < c < b$, where a, b, c are real numbers corresponding to points A, B, C. Here the line

segment AB is said to be divided internally at C, the point of division.

If $AC=3\cdot5$ cm and $CB=4\cdot5$ cm, the ratio $AC:CB=3\cdot5:4\cdot5$ or $7 : 9$. Thus the segment AB is divided internally at C in the ratio $7 : 9$. A is the initial point, B is the terminal point and C is the point of division.

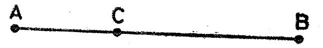


Fig. 3.1

External division.

When the line segment AB is extended to C, C may be still called a point of division of AB. In this case, C is said to divide AB *externally* into two line segments AC and CB or in the ratio $AC : CB$. We still stick to the convention of measuring the line segments from the point of division viz. AC and CB (and not BC). In the case of external division, a change of direction is, therefore, necessary. If $AB=5$ cm and $BC=3$ cm then the ratio $AC : CB=8 : -3$ or $-\frac{8}{3}$.

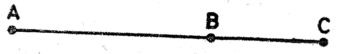


Fig. 3.2

Postulate: **A line segment is divided at one and only one point in a given ratio.**

Definitions. (1) If a, b, c, d are four real numbers such that $a : b=c : d$, d is called the *fourth proportional* to a, b, c.

(2) If $a : b=b : c$ *i.e.*, if $\frac{a}{b}=\frac{b}{c}$ or $b^2=ac$, b is called the mean proportional between a and c and c is called the third proportional to a and b.

Similar definitions hold for line segments. Thus if the line segments AB and AC are divided at D and E so that $AD : DB=AE : EC$, then EC is the fourth proportional to AD, DB, AE. Also if AB and CD are two line segments of lengths p and q units, a segment of length $\sqrt{pq}$ is called a mean proportional between AB and CD.

(3) If a segment AB is divided internally at P and externally at Q in ratios which are numerically equal, the points P and Q are said to divide AB *harmonically* or the four points A, P, B, Q are said to form a *harmonic range.*

Theorem 1

If two triangles have equal altitudes, their areas are proportional to their bases.

Let a, a' units be the lengths of the bases of the two triangles and h units the altitude of each triangle.

The area of a triangle $=\frac{1}{2}$ base $\times$ altitude.

$\therefore$ the ratio of the areas of the triangles here

$$=\tfrac{1}{2}ah : \tfrac{1}{2}a'h = a : a' \text{ or } \frac{a}{a'}.$$

In this context, $\triangle ABC$ means the area of triangle ABC.

Corollary. If two triangles have equal bases, then their areas are proportional to their altitudes.

Exercise 1

1. Find the ratio of $\triangle ABD$ to $\triangle ADC$ where D is a point on the base such that $AD : DC = 1 : 3$.

2. In the above example, H is the midpoint of AD. Find the ratio of $\triangle ABH$ to ABC.

3. In the triangle ABC, BC is divided into four equal parts at the points D, E, F and AB is divided into four equal parts at the points P, Q, R. Find the ratios

$$\frac{\triangle ABD}{\triangle ABC},\ \frac{\triangle ABF}{\triangle ABC},\ \frac{\triangle ABE}{\triangle APC},\ \frac{\triangle AQC}{\triangle ARF}.$$

4. $ABCD$ is a parallelogram and P is a point on the diagonal BD such that $BP : PD = 1 : 3$. Prove that $\triangle APD = \frac{3}{8}$ ▱^{m} $ABCD$.

5. Prove that the areas of the four triangles into which a quadrilateral is divided by the diagonals are proportional.

6. O is any point outside the triangle ABC. If AO intersects BC at D, prove that $\triangle AOB : \triangle AOC = BD : DC$.

Theorem 2 (a)

If a line parallel to one side of a triangle intersects the other two sides in distinct points, the segments of these sides are proportional.

Given: A line l parallel to the side BC of the triangle ABC and cutting the sides AB, AC in D, E respectively.

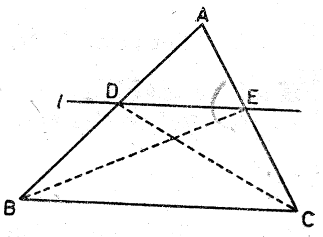


Fig. 3.3

To prove that: $AD : DB = AE : EC$ or $\dfrac{AD}{DB} = \dfrac{AE}{EC}$

Construction: Join BE and CD.

Proof: By Theorem 1, $\dfrac{\triangle AED}{\triangle DEB} = \dfrac{AD}{DB}$

and $\dfrac{\triangle ADE}{\triangle DEC} = \dfrac{AE}{EC}$

But $\triangle DEB = \triangle DEC$, since $DE \parallel BC$.

$$\therefore \frac{AD}{DB} = \frac{AE}{EC}.$$

Note. Write the areas $\triangle AED$ etc. in the same order as the segments AD etc.

Corollary. We have also $\dfrac{AD}{AB} = \dfrac{AE}{AC}$ and $\dfrac{BD}{AB} = \dfrac{CE}{AC}$.

This follows from the properties of ratios.

[If $\dfrac{a}{b} = \dfrac{c}{d}$, $\dfrac{b}{a} = \dfrac{d}{c}$ and $\dfrac{a}{c} = \dfrac{b}{d}$ as in either case $ad = bc$.

By adding 1 to each side of the ratios, $\frac{a}{b}+1=\frac{c}{d}+1$

or $\frac{a+b}{b}=\frac{c+d}{d}$ $\quad\therefore \frac{b}{a+b}=\frac{d}{c+d}$.]

Theorem 2 (b) (Converse Theorem)

If a line divides two sides of a triangle proportionally, it is parallel to the third side.

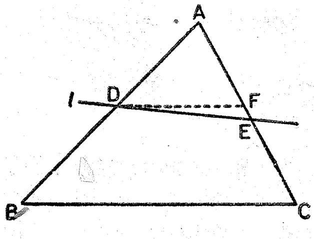


Fig. 3.4

Given: The line l meets AB and AC in D and E such that $\frac{AD}{DB}=\frac{AE}{EC}$

To prove that: $l \parallel BC$.

Proof: If DE is not parallel to BC, let DF be drawn through D, parallel to BC. [DF exists by Euclid's Postulate.]

Then, by Theorem 2(a), $\frac{AD}{DB}=\frac{AF}{FC}$

$\therefore \frac{AD}{DB}=\frac{AE}{EC}=\frac{AF}{FC}$

But there is only one point which divides the segment AC in a given ratio.

∴ F and E must coincide.

i.e., $DE \parallel BC$.

The student should also write down the proofs of Theorems 2(a) and 2(b) by taking l as a line which cuts AB and AC externally.

Construction 1

To divide a given segment (internally or externally) in a given ratio.

Let AB be a given segment and p, q two real numbers.

Draw any other line l through A, and cut off segments AP and PQ to be of lengths equal to p and q. Join BQ and through P draw $PX \parallel BQ$, to cut AB in X. Then X is the required point dividing AB in the ratio $p:q$.

The figures below show three cases, according as $\frac{p}{q}$ is positive, negative and numerically > 1, or negative and numerically < 1. The construction is the same in all the cases.

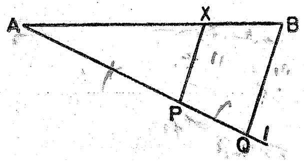


Fig. 3.5(a)

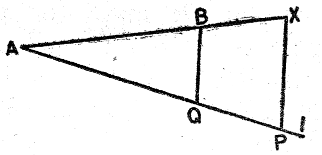


Fig. 3.5(b)

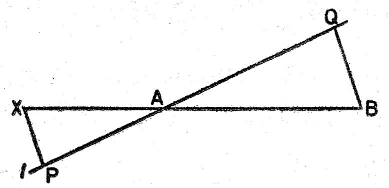


Fig. 3.5(c)

In Fig. 3.5(a), p and q are both positive; AP and PQ are in the same direction. ∴ AX and XB are in the same direction. In Fig. 3.5(b), p and q are of opposite signs. We take $AP=p$ along l, and PQ in the *reverse direction*, equal to q units numerically. Q will be between A and P. In Fig. 3.5(c), p and q are of opposite signs. We take $AP=p$ along l, and PQ in the reverse direction, equal to q units numerically. Q will be a point external to AP.

When $\frac{p}{q}$ is negative and numerically > 1, the point X will be on AB produced. If $\frac{p}{q}$ is negative and numerically < 1, X will be on BA produced.

Construction 2

To construct the segment which is the fourth proportional to three given segments.

Along a line l, mark the segments OA and AB equal to the first two segments. Along another line m through O, mark the third segment OC. Join AC and draw BD parallel to AC to cut m in D. Then CD is the required 4th proportional.

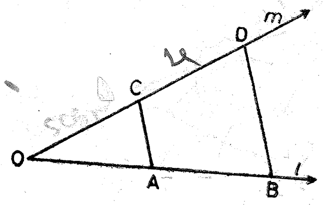


Fig. 3.6

Proof: By the Triangle Proportionality Theorem 2(a), $OA : AB = OC : OD$ $[\because CA \parallel DB]$.

Construction 3

To construct the third proportional to two given segments.

Here on the line l, mark off segments OA, AB equal to the given segments. Draw another line m and mark off a point C on it so that $OC = AB$. Through B draw a line parallel to CA to cut m at D.

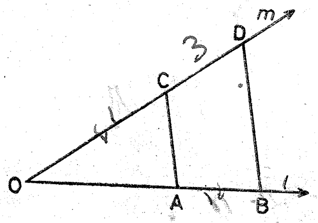


Fig. 3.7

Since $AC \parallel BD$, $\frac{OA}{AB} = \frac{OC}{CD}$.

$\therefore \frac{OA}{AB} = \frac{AB}{CD}$ $\because OC = AB$ by construction.

$\therefore AB^2 = OA \cdot CD$

$\therefore CD$ is the third proportional to AB and OA.

Exercise 2

1. By using Theorem 2(a), prove that the midpoint of the hypotenuse of a right-angled triangle is equi-distant from the three vertices. [*Hint.* Draw through the midpoint a line segment parallel to one of the sides containing the right angle.]

2. By using Theorem 2(b) prove that the line joining the middle points of two sides of a triangle is parallel to the third side.

3. Three parallel lines are intersected by two transversals in A, B, C and L, M, N respectively. Prove that $AB : BC = LM : MN$.

4. In a $\square^m$ $ABCD$, P and Q are the middle points of AD and BC respectively. BP and DQ intersect AC at M and N respectively. Prove that BP and DQ trisect AC.

5. In $\triangle ABC$, D is a point on AB and $DE \parallel BC$ cutting AC at E. $EF \parallel AB$ cutting BC at F. Prove that $AB : BD = BC : CF$.

6. $ABCD$ is a trapezium with $AB \parallel DC$, and the diagonals AC and BD meet at E. Prove that $AE : EC = BE : ED$.

7. $ABCD$ is a quadrilateral. P is any point on AB. Draw $PQ \parallel BC$ meeting AC at Q. Draw $QR \parallel AD$ meeting CD at R. Prove that $DQ : RC = AP : PB$.

8. ABC, $A'BC$ are two triangles on the same side of BC. L is any point on BC. LM drawn parallel to AB cuts AC in M. LN drawn parallel to $A'B$ cuts $A'C$ in N. Prove that MN is parallel to AA'.

9. *ABCD* is a trapezium in which *AB* is parallel to *DC*. *BC* and *AD* are divided in the same ratio at *L* and *M*. Prove that *LM* is parallel to *AB*. Prove the converse also.

10. In a $\triangle ABC$, *AX* is drawn perpendicular to the internal bisector of the angle *B*. Show that the line through *X* parallel to *BC* will bisect *AB* and *AC*.

11. Draw a line 6 cm long and divide it in the ratio (1) 2 : 3, (2) 2 : −3, (3) 2 : −1.

12. Divide a line 8 cm long internally and externally in the ratio 3 : 4.

13. If *a* and *b* are the lengths of two segments, construct geometrically segments whose lengths are (i) $\frac{1}{a}$, (ii) $\frac{b^2}{a}$.

14. Construct a triangle of perimeter 10 cm whose sides are in the ratio 1 : 2 : 3.

15. In this figure *ABCD* is any quadrilateral and *X*, *Y* and *Z* are points on *DA*, *DB*, *DC* such that $XY \parallel AB$ and $YZ \parallel BC$. Prove that $XZ \parallel AC$.

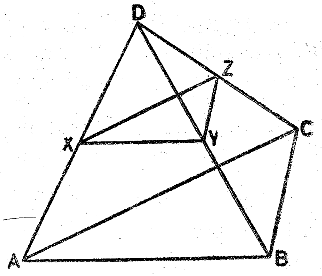


Fig. 3.8

Similar figures.

Two geometrical figures are said to be *similar* if they have the same shape *i.e.* if one is the magnification of the other.

A.V. Adhav. Near Bus Stand. Nellikeri. Kumta- N.K.

If a picture of a quadrilateral *ABCD* is projected on a screen in a plane parallel to the plane of the slide, the image *A′B′C′D′* will be a uniform magnification of the original quadrilateral. *ABCD* may be considered a uniform reduction of the quadrilateral *A′B′C′D′*. Quadrilaterals *ABCD* and *A′B′C′D′* are, in such cases, said to be similar. They have the same shape but not the same size. That is why they have the same internal angles. Also the lengths of sides are proportional.

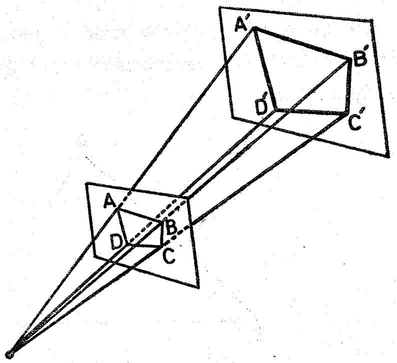


Fig. 3.9

Two figures are similar if (1) corresponding angles are equal and (2) corresponding sides are proportional.

The following quadrilaterals are similar.

In the general case, both conditions are necessary. Thus a rectangle and a square have equal angles, but they are not similar. A square and a rhombus have corresponding sides proportional, but they are not similar.

In the case of two similar triangles, however, one of these conditions is sufficient. We can prove that one condition follows from the other. These are proved below as Theorems 3 and 4.

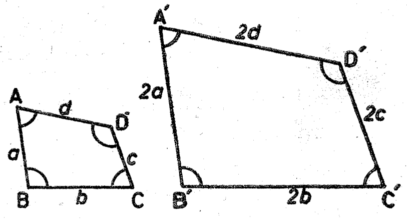


Fig. 3.10

Theorem 3

If two triangles are equiangular, their corresponding sides are proportional and hence the triangles are similar.

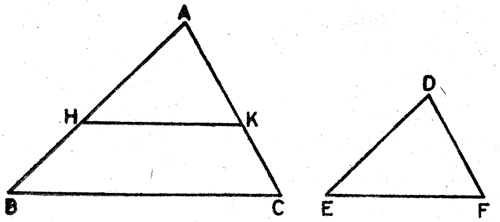


Fig. 3.11

Given: ABC, DEF are two $\triangle^s$ having $\hat{A}=\hat{D}$, $\hat{B}=\hat{E}$, $\hat{C}=\hat{F}$.

To prove that: $AB : BC : CA = DE : EF : FD$.

Construction: Take H on AB and K on AC (produced if necessary) such that $AH=DE$ and $AK=DF$.

Proof: $\triangle AHK \equiv \triangle DEF$, since $AH=DE$ and $AK=DF$ and $\hat{A}=\hat{D}$.

$\therefore A\hat{H}K = D\hat{E}F$ and $A\hat{K}H = D\hat{F}E$

$\therefore A\hat{H}K = A\hat{B}C$ $\quad \therefore HK \parallel BC$

$\therefore \dfrac{AH}{AB} = \dfrac{AK}{AC}$ $\quad \therefore \dfrac{DE}{AB} = \dfrac{DF}{AC}$ or $AB : AC = DE : DF$

Similarly repeating the proof at the vertices B and E, we have

$\dfrac{DE}{AB} = \dfrac{EF}{BC}$ or $AB : BC = DE : EF$.

$\therefore DE : EF : FD = AB : BC : CA$, as DF and FD or AC and CA are numerically the same.

Hence, by definition, the $\triangle^s$ are similar.

Theorem 4

If two triangles have the three sides of one proportional to the three sides of the other, taken in order, the triangles are equiangular, and are therefore similar.

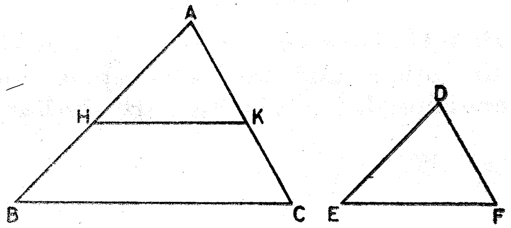


Fig. 3.12

Given: In the $\triangle^s$ ABC, DEF,

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}.$$

To prove that: $\hat{A}=\hat{D}$, $\hat{B}=\hat{E}$, $\hat{C}=\hat{F}$.

Construction: From AB, AC (produced if necessary), cut off $AH=DE$ and $AK=DF$. Join HK.

Proof: Now $\dfrac{AB}{DE} = \dfrac{AC}{DF}$. $\quad \therefore \dfrac{AB}{AH} = \dfrac{AC}{AK}$ (by construction)

$\therefore HK \parallel BC$. (Theorem 2(*b*))

$\therefore \triangle$s AHK and ABC are equiangular.

$\therefore$ By Theorem 3, $\dfrac{AB}{AH} = \dfrac{BC}{HK}$

But $\dfrac{AB}{AH} = \dfrac{AB}{DE} = \dfrac{BC}{EF}$

$\therefore \dfrac{BC}{HK} = \dfrac{BC}{EF}$ $\quad \therefore EF = HK$.

Now in △s AHK, DEF, $AH=DE$, $AK=DF$, $HK=EF$.

∴ The △s are congruent.

∴ $A\hat{H}E=\hat{E}$, $A\hat{K}H=\hat{F}$, and $\hat{A}=\hat{D}$

∴ $\hat{B}=\hat{E}$, $\hat{C}=\hat{F}$, $\hat{A}=\hat{D}$.

Theorem 5

If two triangles have one angle of the one equal to an angle of the other, and the sides about these equal angles proportional, the triangles are similar.

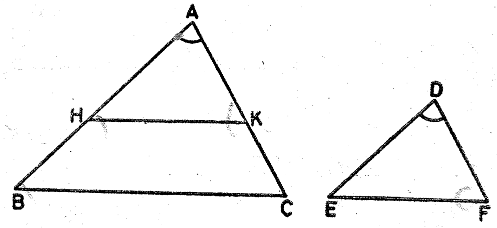


Fig. 3.13

Given: In the two △s ABC, DEF,

$$\hat{A}=\hat{D}, \text{ and } \frac{AB}{DE}=\frac{AC}{DF}.$$

To prove that the △s are similar.

Construction: From AB, AC produced if necessary, cut off $AH=DE$, $AK=DF$. Join HK.

Proof: The △s AHK, DEF are congruent ($\hat{A}=\hat{D}$, $AH=DE$, $AK=DF$)

∴ $A\hat{H}K=D\hat{E}F$, $A\hat{K}H=D\hat{F}E$

Also, $\dfrac{AB}{AH}=\dfrac{AC}{AK}$, since $AH=DE$, $AK=\hat{D}F$

∴ $HK \parallel BC$

∴ $A\hat{H}K=\hat{B}$, $A\hat{K}H=C$.

∴ $\hat{B}=\hat{E}$, and $\hat{C}=\hat{F}$.

We have also $\hat{A}=\hat{D}$. ∴ △s ABC, DEF are equiangular and hence similar.

Theorem 6

If two triangles have one angle of the one equal to one angle of the other, and the sides about another angle of the first triangle proportional to the sides about another angle of the other, then the third angles are either equal or supplementary; in the former case, the triangles are similar.

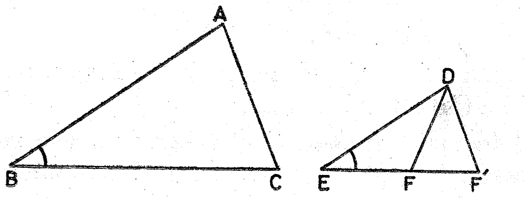


Fig. 3.14

Given: In △s ABC, DEF,

$$\hat{B}=\hat{E} \text{ and } \frac{AB}{DE}=\frac{AC}{DF}.$$

To prove that

either (1) $C=F$

or (2) $C+F=180°$.

Case I. Suppose $\hat{A}=\hat{D}$.

Then since $\hat{B}=\hat{E}$

the third angles C and F are equal.

The △s are equiangular and hence similar.

Case II. Suppose $\hat{A} \neq \hat{D}$.

Through D draw DF' such that $\hat{A} = E\hat{D}F'$.

$\therefore$ $\triangle$s ABC, DEF' are equiangular and hence similar

$$\therefore \frac{AB}{DE} = \frac{AC}{DF'} \text{ and } \hat{C} = \hat{F}.$$

But by hypothesis, $\dfrac{AB}{DE} = \dfrac{AC}{DF}$

$$\therefore DF = DF'$$

$$\therefore D\hat{F}F' = D\hat{F}'F$$

$$\text{But } D\hat{F}E = 180° - D\hat{F}F'$$

$$\therefore \hat{D} = 180° - D\hat{F}'F$$

$$= 180° - \hat{C}.$$

This case can arise only if one of the triangles, has an obtuse angle, while the other has not.

Cor. **If in two right-angled triangles, the ratio of the hypotenuses is equal to the ratio of two corresponding sides, the triangles are similar.**

For the other angles of the two triangles are all acute, and case *II* above cannot arise.

Theorem 7

In a right-angled triangle, if a perpendicular is drawn from the vertex of the right angle to the hypotenuse, the triangles on each side of the perpendicular are similar to the whole triangle, and to one another.

ABC is a $\triangle$ rt.-angled at *A*. *AD* is perpendicular to *BC*.

To prove that the two $\triangle$s so formed *ABD* and *ACD* are similar to each other and to $\triangle ABC$.

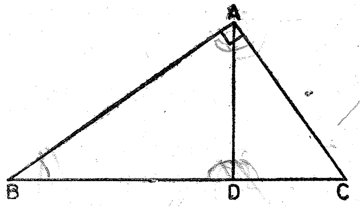


Fig. 3.15

Proof: $D\hat{A}C = A\hat{B}D$, being both complements of BAD

$A\hat{D}C = A\hat{D}B$

$\therefore$ $\triangle$s ADC and BDA (with vertices in this order) are similar.

Similarly, in $\triangle$s DAC, ABC,

$A\hat{D}C = B\hat{A}C$, $D\hat{A}C = A\hat{B}C$, and angle C is common.

$\therefore$ $\triangle$s are similar.

Cor. 1*:* Taking ratios of corresponding sides of $\triangle$s ADC, BDA.

$$\frac{AD}{DC} = \frac{BD}{AD} \qquad \therefore \mathbf{BD \cdot DC = AD^2}$$

or *AD is the mean proportional between the segments BD and DC.*

Again, from $\triangle$s ADC, BAC,

$$\frac{AC}{DC} = \frac{BC}{AC} \quad \text{or} \quad \mathbf{CA^2 = CB \cdot CD}$$

Similarly, from $\triangle$s ABD, CBA, we can prove

$$\mathbf{BA^2 = BD \cdot BC}$$

Cor. 2*:* Adding the above results,

$$AB^2 + AC^2 = BC \cdot (BD + DC) = BC^2.$$

This is Bhaskara's (A.D. 1150) proof of Pythagoras' Theorem.

Construction 4

To find the mean proportional between two given segments.

Construction: Place the segments side by side on the same line. Let them be *BD* and *DC* as in the figure above. Construct a semi-circle on *BC* as diameter and draw $DA \perp BC$ to meet the semi-circle at *A*.

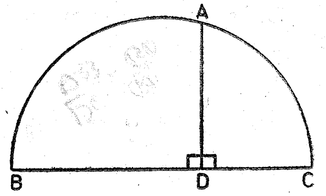


Fig. 3.16

Then *AD* is the required mean proportional, for $AD^2 = BD \cdot DC$. (Vide Cor. 1, Theorem 7).

Exercise 3

1. PQR, XYZ are two similar triangles in which $QR=7$ cm, $RP=$cm, $PQ=5$ cm, $P=120°$, $YZ=10\cdot5$ cm. Write down the values of XY, XZ and X.

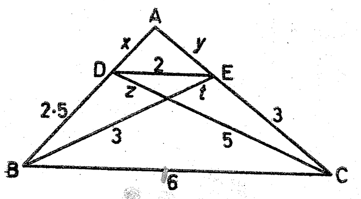
2. Are the following measurements correct?

(1) ABC, DEF are similar triangles in which
$BC=3$, $AB=5$, $B=120°$, $EF=6$, $DE=5$, $E=120°$.

(2) ABC, DEF are similar triangles in which
$BC=2$, $AB=5$, $A=30°$. $DE=20$, $EF=8$, $D=30°$.

3. Prove that the triangle whose sides measure 5·7 cm, 7·6 cm, 9·5 cm is right-angled.

4. In the annexed figure, calculate the lengths of the segments marked x, y, z, t.



Here $DE \parallel BC$.

Fig. 3.17

5. If two triangles have the sides of one respectively parallel to the sides of the other, the triangles are similar.

6. If two triangles have the sides of one respectively perpendicular to the sides of the other, the triangles are similar.

7. In two similar triangles, the corresponding medians are in the same ratio as corresponding sides.

8. Prove the result of Ex. 7 above for

(i) corresponding altitudes, (ii) the circum-radii, (iii) the radii of the inscribed circles, (iv) bisectors of corresponding angles.

9. In any triangle, the median bisecting a side bisects any segment drawn parallel to that side and terminated by the other two sides.

✓**10. The rectangle contained by the segments of two intersecting chords of a circle are equal.**

[If AOB, COD are two chords, $\triangle$s AOD, COB are similar.]

✓**11. If from an external point, a secant and a tangent are drawn to a circle, the square on the tangent is equal to the rectangle contained by the segments of the secant.**

[Let PT be a tangent, and PAB a secant drawn through P. Prove $\triangle$s PTA and PBT to be similar.

Also prove the result as a limiting case of the result of Ex. 10.]

12. If two segments AB and CD intersect at O, and if $AO \cdot OB = CO \cdot OD$, prove that A, B, C, D are concyclic. (Converse of Ex. 10; use reductio ad absurdum method.)

13. If two segments PAB and PT are such that $PT^2=PA \cdot PB$, prove that PT touches the circle TAB. (Converse of Ex. 11; use reductio ad absurdum method.)

14. The tangent at a point P to a circle whose centre is O cuts a pair of parallel tangents at Q and R. Prove that $OP^2=PQ \cdot PR$.

15. OP, OQ are tangents to a circle, touching it at P and Q. R is any other point on the circle, and RL, RM, RN are drawn perpendicular to OP, OQ, PQ respectively. Prove that $RL \cdot RM = RN^2$.

16. The tangent at A to the circumcircle of $\triangle ABC$ meets BC produced in D. Prove that $\dfrac{BD}{CD}=\dfrac{AB^2}{AC^2}$. [$\triangle$s ACD, BAD are similar.]

17. Prove that the common tangent to two circles touching externally is a mean proportional between their diameters.

18. If two circles intersect at P and the tangents at P form chords PQ, PR, prove that PQ, PR are proportional to the diameters of the circles.

19. BY, CZ are drawn perpendicular to the tangent at A to the circumcircle of $\triangle ABC$. AX is the altitude on BC. Prove that $AX^2=BY \cdot CZ$.

20. Through the vertex A of a parallelogram $ABCD$, a line is drawn to cut BD, CD and BC at P, Q, R respectively. Prove that $AP^2 = PQ \cdot PR$.

21. A circle touches the side AB of $\triangle ABC$ at A, and passes through C. Another circle touches AC at A and passes through B. The common chord of the circles meets BC at P. Prove that $\frac{BP}{PC} = \frac{AB^2}{AC^2}$.

22. The rectangle contained by two sides of a triangle is equal to the rectangle contained by the altitude and the diameter of the circumcircle.

[Let ABC be the $\triangle$, and AD the altitude through A. Let AE be the diameter through A of the circumcircle. Prove that $\triangle$s ABE and ADC are similar.]

23. Prove that the internal bisector of an angle of a triangle divides the opposite side in the ratio of the other two sides. Prove also the converse.

[If AD is the bisector of $\hat{A}$, draw $CE \parallel AD$ to meet BA produced at E. Prove $AE = AC$].

24. Prove that the bisector of the external angle of a triangle divides the opposite side externally in the ratio of the other two sides. Prove also the converse.

25. ABC is a triangle right-angled at A. AD is perpendicular to BC. The internal bisector of $B\hat{A}C$ cuts BC in L. Prove that $\frac{BD}{DC} = \frac{BL^2}{LC^2}$. [Use Ex. 23.]

26. Two lines OAB, OCD cut a circle at A, B, C, D. A line OL parallel to BC meets DA produced at L. Prove that $OL^2 = LA \cdot LD$.

27. Find the square roots of 2 and 15 geometrically, by the method of mean proportional. Hence again find the square root of $2\sqrt{2}$.

Drop

Theorem 8

The areas of similar triangles are proportional to the squares of the lengths of corresponding sides.

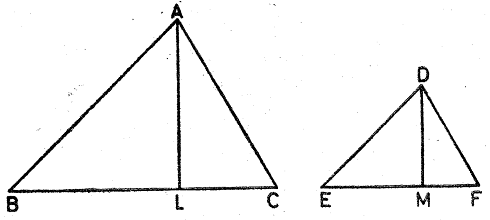


Fig. 3.18

ABC, DEF are two similar $\triangle$s. AL, DM are altitudes from A and D.

$\triangle ABC = \frac{1}{2} BC.AL$

$\triangle DEF = \frac{1}{2} EF \cdot DM$

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{BC}{EF} \cdot \frac{AL}{DM}$$

But $\triangle$s ABL, DEM are similar. $[\hat{B} = \hat{E};\ \hat{L} = \hat{M} = 90°]$

$$\therefore \frac{AL}{DM} = \frac{AB}{DE}$$

$$= \frac{BC}{EF}, \text{ since } \triangle\text{s } ABC, DEF \text{ are similar.}$$

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{BC}{EF} \cdot \frac{BC}{EF} = \frac{BC^2}{EF^2}. \quad \text{Since } \frac{BC}{EF} = \frac{AB}{DE} = \frac{AC}{DF},$$

$$= \frac{AB^2}{DE^2} = \frac{AC^2}{DF^2}.$$

Exercise 4

1. ABC is any triangle and AB is divided internally at a point P in the ratio of 3 : 2. Draw $PQ \parallel BC$ to meet AC at Q. Calculate the ratio of $\triangle APQ : \triangle ABC$.

2. If, in example 1, P is the point of external division of AB in the given ratio, state the ratio of $\triangle APQ : \triangle ABC$.

3. Two triangles ABC and PQR have their areas in the ratio of 16 : 49. If $PQ = 3{\cdot}5$ cm, $QR = 2{\cdot}8$ cm, calculate the corresponding sides AB and BC of the other triangle.

4. In $\triangle ABC$, medians BD and CE meet at F. Calculate the ratio of the areas of $\triangle DEF$ and $\triangle BCF$.

5. In $\triangle ABC$, a line $l \parallel BC$ intersects AB and AC at D and E respectively. If the areas of trapezium $DECB = \frac{7}{16}\triangle ABC$, find $AD : DB$.

6. The sum of a pair of the corresponding sides of two similar triangles is 12 cm. If the ratio of the areas of the two triangles is 3 : 1, show that the lengths of the side of the first triangle referred to above is $18 - 6\sqrt{3}$ cm.

7. ABC, PQR are two triangles in which $\hat{A} = \hat{P}$. Prove that $ABC : PQR = AB \cdot AC : PQ \cdot QR$.

8. P and Q are points in AB of $\triangle ABC$ such that the lines drawn parallel to BC from them divide the $\triangle ABC$ into three equal parts. Prove that $AP : AQ : AB = 1 : \sqrt{2} : \sqrt{3}$.

9. Tangents at any points A, B to a circle, centre O, meet at T. Prove that $\triangle ATB : \triangle OAB = AT^2 : AO^2$.

10. Prove that the areas of similar triangles are proportional to (i) the squares on corresponding altitudes, (ii) the squares on corresponding medians, (iii) the squares on the radii of their circumscribed circles and (iv) the squares on the bisectors of corresponding angles.

CHAPTER 4

Simple Solids

16. The parallelepiped.

This is a solid formed by three pairs of planes, which are parallel, two by two, as in Fig. 4.1. It will be readily proved that each face of the solid is a parallelogram, and that the parallelograms forming opposite faces are congruent. If the parallelograms are all rectangles, the figure is called a *cuboid*. If they are all squares, the figure is a *cube*.

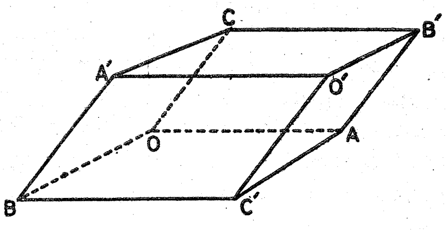


Fig. 4.1

OO', AA', BB', CC' are the diagonals of the parallelepiped. Since OA is equal and parallel to $O'A'$, the figure $OA'O'A$ is a parallelogram. Hence its diagonals OO' and AA' bisect each other. Similarly, *all the four diagonals intersect and bisect one other.*

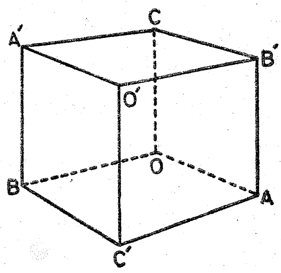


Fig. 4.2

If the figure is a cuboid, whose edges OA, OB, OC are respectively of lengths a, b, c, then

$$OO'^2 = OC'^2 + C'O'^2 = OA^2 + AC'^2 + C'O'^2$$
$$= a^2 + b^2 + c^2$$

Hence, *the diagonals of a cuboid are all equal.*

17. The pyramid and the tetrahedron. If the points on the sides of a plane polygon are all joined to a point O outside the plane, the solid formed is called a pyramid. O is its *vertex*, and the polygon the *base*.

When the base is a triangle, the pyramid is called *tetrahedron*. A tetrahedron is thus a triangular pyramid. It can also be described as the figure obtained from four points, all of which do not lie in the same plane. It has four faces, four vertices and six edges.

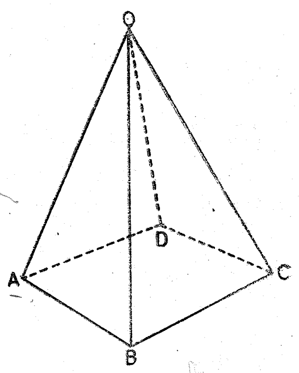


Fig. 4.3

If the base of a pyramid is a regular polygon, it is called a *regular pyramid*. In a *regular tetrahedron*, all the faces are equilateral triangles.

If $ABCD$ is a tetrahedron, the edges two by two are skew. These are called opposite edges. Thus AB and CD, BC and AD, AC and BD are opposite edges.

Calculate the dihedral angle between two faces of a regular tetrahedron.

If g be the centroid of $\triangle BCD$, and L the midpoint of BC, then $AL \perp BC$, $LD \perp BC$, and Ag is normal to the base $\triangle BCD$. The required dihedral angle between the faces BCD and ABC is $A\hat{L}D$.

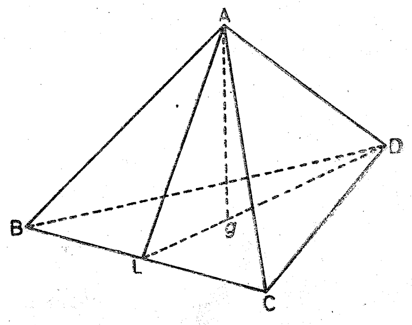


Fig. 4.4

We have $\cos A\hat{L}g = \dfrac{Lg}{AL} = \frac{1}{3}$, since $AL = LD$ and $Lg = \frac{1}{3}LD$.

The dihedral angle is therefore θ where $\cos\theta = \frac{1}{3}$.

Exercise 1

1. For a regular tetrahedron, calculate (1) the angle made by a slant edge with the base, (2) the angle between two opposite edges.

2. A square pyramid has its base a square of side 5 cm, and the length of each slant edge is 8 cm. Calculate

(i) the height of the pyramid
(ii) the angle between a slant face and the base
(iii) the angle between the base and a slant face.

Theorem 13

The lines joining the middle points of opposite sides of a tetrahedron are concurrent and bisect one another.

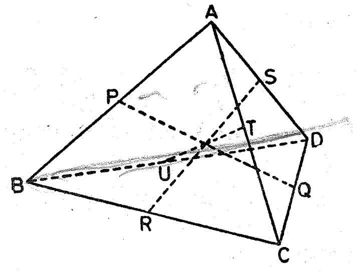


Fig. 4·5

Let P, Q, R, S, T, U be the middle points of (AB, CD), (BC, AD) and (AC, BD) respectively of a tetrahedron $ABCD$.

From $\triangle ABD$, $PS \parallel BD$ and equal to $\frac{1}{2}BD$

From $\triangle BCD$, $RQ \parallel BD$ and equal to $\frac{1}{2}BD$

$\therefore$ $PSQR$ is a parallelogram.

$\therefore$ The diagonals PQ and RS intersect and bisect each other.

Similarly $PTQU$ is a parallelogram and hence PQ and UT intersect and bisect each other. Thus PQ, RS, UT intersect and bisect one another.

Theorem 14

The lines joining the vertices of a tetrahedron to the centroids of the opposite faces are concurrent at a point which divides each of these lines in the ratio 3 : 1.

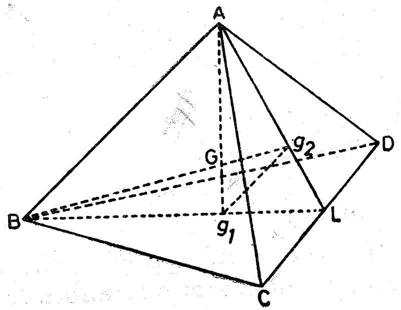


Fig. 4.6

Let L be the middle point of CD, and g_1, g_2 the centroids of $\triangle^s BCD, ACD$ respectively of a tetrahedron $ABCD$. Then g_1 lies on BL, g_2 lies on AL, and $Bg_1 : g_1L = Ag_2 : g_2L = 2 : 1$.

$\therefore \dfrac{AL}{g_2L} = \dfrac{BL}{g_1L} = \frac{3}{1}$. The lines Ag_1 and Bg_2 both lie in the plane ABL, and hence intersect in some point G.

Now $g_1g_2 \parallel AB$.

$\therefore$ $\triangle$s AGB and g_1Gg_2 are similar.

$$\therefore \frac{AG}{Gg_1} = \frac{BG}{Gg_2} = \frac{AB}{g_1g_2}.$$

$$\text{The last ratio } = \frac{BL}{g_1L}, \text{ since } \triangle^s\ ABL,\ g_1g_2L \text{ are similar.}$$

$$= \tfrac{3}{1}.$$

Thus G divides Ag_1 in the ratio 3 : 1. Similarly G divides the lines joining the other vertices of the tetrahedron to the centroids of the opposite faces in the ratio 3 : 1. Hence the theorem.

G is called the *centroid of the tetrahedron.*

18. The *prism* is the solid formed by joining the corresponding sides of two polygons whose sides and angles are correspondingly equal, lying in parallel planes in such a way that the corresponding sides are parallel and in the same sense. If the lines joining corresponding vertices of the polygons are normal to their planes, the prism is a *right* prism. (Fig. 4.7)

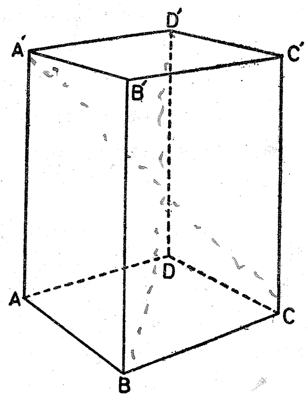


Fig. 4.7

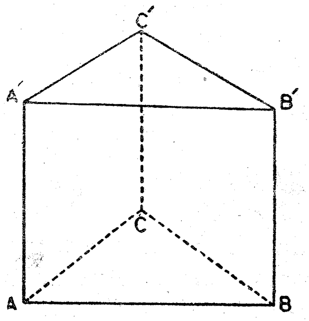


Fig. 4.8

In particular, if the polygons are triangles, we have the *triangular prism.* The lines joining the corresponding vertices of the two triangles are the *edges* of the prism.

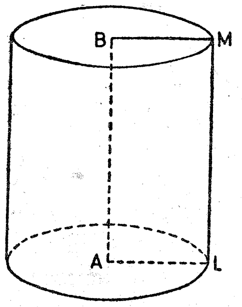


Fig. 4.9

Suppose in a right prism, the number of sides of the two polygons is increased indefinitely, so that the length of each side becomes as small as we like. The polygons ultimately become two congruent closed curves in parallel planes, and the prism is now called a *cylinder.* If the curves are circles, the cylinder is called a *right circular cylinder.*

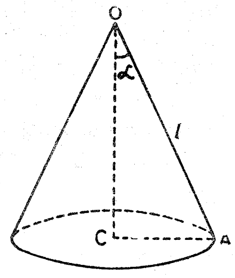


Fig. 4.10

In a similar manner, by increasing indefinitely the number of sides in the base of a pyramid such that each side becomes as small as we like, we obtain a *cone*. A cone is thus the figure obtained by joining the points of a plane closed curve to a vertex outside the plane. When the base is a circle, and the vertex lies on the normal to the base through its centre, the cone is called a *right circular cone.*

The right circular cylinder and the right circular cone can also be considered as surfaces of revolution.

The figure obtained by rotating one side LM of a retangle $ABML$ about the parallel side AB is the right circular cylinder (Fig. 4.9). The different positions that the line LM takes during the course of the revolution are called generating lines or *generators*. Any point on LM describes a circle—the *cross-section* of the cylinder.

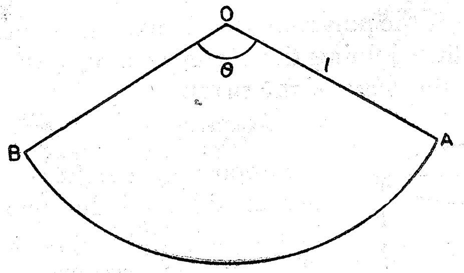


Fig. 4.11

The figure obtained by rotating the hypotenuse OA of a right-angled triangle OCA (see Fig. 4.10), about one side OC is the right circular cone. The different positions of the line OA during the rotation are the *generators* of the cone. Any point on OA describes a circle—the cross-section of the cone. The generators make a constant angle with the axis OC. This angle is the semi-vertical angle of the cone.

Consider the cylinder in Fig. 4.9 to be cut along the generator LM, and the cylinder opened out into a plane. We obtain a rectangle for wich one side is LM, and the other side has its length equal to the circumference of the circular base.

It follows that the curved surface of the cylinder is equal to the area of this rectangle. If r is the radius of the circular base, this area$=LM\times 2\pi r$.

$\therefore$ **curved surface of the right circular cylinder=area of base × height.**

In a similar manner, if the right circular cone in Fig. 4.10 is cut along OA and opened out, we obtain a sector of a circle (Fig. 4.11). The arc of this sector is equal in length to the circumference of the base of the cone.

Let r be the radius of the circular base, and h the height of the cone. The radius of the sector obtained by opening out the cone is l, where $l=OA=$the *slant height* of the cone. If θ is the angle of the sector,

$$l\theta=2\pi r.$$

The curved surface of the cone is equal to the area of the sector. Now the area of a sector of a circle, radius r, and subtending an angle α at the centre is known to be $\frac{1}{2}r^2\alpha$.

$$\therefore \text{ curved surface}=\tfrac{1}{2}l^2\theta=\tfrac{1}{2}l^2\cdot\frac{2\pi r}{l}=\pi rl.$$

Curved surface of a right circular cone$=\pi rl$, where r is the radius of base, and l the slant height.

19. Locus Problems. (1) A point P is at a constant distance from a given line l. What is the locus of P(i) in a given plane through l (ii) in three-dimensional place.

Ans. (i) A parallel line

(ii) A right circular cylinder with l as axis.

(2) A line m meets a fixed line l at a fixed point and makes a constant angle with it. What is the locus of m in space?

Ans. The surface of a right circular cone, with l as axis.

(3) What is the locus of a point P whose distance from any point on the circumference of a circle is constant?

Ans. The normal through the centre of the circle to its plane.

(4) What is the locus of a point P whose distance from a given line l is constant, and the foot of the perpendicular from P on l is a fixed point?

[If P_1, P_2, ... are the different positions of P, and O is the fixed point, OP_1, OP_2, ... are all coplanar and of equal length. Hence the locus of P is a circle with centre O, viz. the cross-section through O of the right circular cylinder with l as axis.]

20. Mensuration formulae.

Volume of a prism = Area of base × height
,, ,, a pyramid = $\frac{1}{3}$ area of base × height
,, ,, a cylinder = area of base × height
,, ,, a cone = $\frac{1}{3}$ area of base × height.

21. The Sphere.

Theorem 15

Any plane section of a sphere is a circle.

Let O be the centre of the sphere, and P any point on the section. Draw OC normal to the plane of the section, from O.

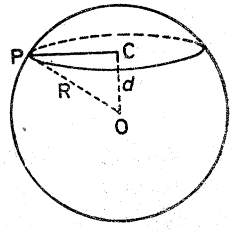


Fig. 4.12

$\therefore O\hat{C}P=90°$, and $OP=R$, the radius of the sphere.

Also let $OC=d$=length of the perpendicular from O to the given plane
= a constant

$\therefore CP=\sqrt{R^2-d^2}$=a constant.

$\therefore$ the locus of P is a circle in the given plane with C as centre and radius $=\sqrt{R^2-d^2}$.

The section is therefore a circle.

Special cases. (1) When $d=0$, the plane of the section passes through the centre of the sphere, and $\sqrt{R^2-d^2}$ is maximum and equal to R, the radius of the sphere. We get the maximum radius for the section, when the section passes through the centre of the sphere. The circle is then called a *great circle*. A great circle is the section of a sphere by a plane passing through the centre of the sphere.

(2) When the section does not pass through the centre of the sphere, $d \neq 0$. The circle obtained is called a small circle.

(3) When $d=R$, the radius of the section is zero. The circle becomes a mere point. The plane is then the tangent plane at this point.

Three non-collinear points determine a plane. To construct a great circle we should therefore be given two points other than the centre of the sphere. These three points form a plane which cuts off the sphere in a great circle. Thus, one and only one great circle can be drawn so as to pass through two given points on the sphere, which are not diametrically opposite. But if the two points are diametrically opposite, they and the centre lie on a line and do not determine a plane. Hence one more point on the sphere is required to determine the great circle. Hence an infinite number of great circles can be drawn through two diametrically opposite points on a sphere.

Mensuration formulae:

Volume of a sphere $=\frac{4}{3}\pi r^3$,
Area of the surface of a sphere $=4\pi r^2$.

Exercise 2

1. Prove that the middle points of four parallel edges of a parallelepiped form a parallelogram.

2. Prove that the four diagonals of a cube are equally inclined to one another.

3. If the diagonals of a cube are equal to a, find a side of the cube.

4. Prove that the sum of the squares on the diagonals of a parallelepiped is equal to the sum of the squares on its edges.

5. Prove that the S.D. between an edge of a cube and a diagonal which does not meet it is the join of their midpoints.

6. If each edge of a tetrahedron is equal to the opposite edge, prove that the shortest distance between two opposite edges is the line joining their middle points.

7. Prove that the S.D. between two opposite edges of a regular tetrahedron is equal to half the diagonal of the square on an edge.

8. OA, OB, OC are three mutually perpendicular lines of lengths a, b, c respectively. Prove that

(i) the area of $\triangle ABC = \frac{1}{2}\sqrt{a^2b^2+b^2c^2+c^2a^2}$

(ii) length of perpendicular from O on plane ABC

$$=\frac{abc}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}.$$

[*Hint.* Let $OL \perp AB$, then $CL \perp AB$. Calculate OL and CL. To prove (ii), express the volume of the pyramid O, ABC in two different ways, taking C as vertex, and O as vertex.]

9. In any tetrahedron, the line joining the middle points of a pair of opposite edges is perpendicular to the common perpendicular to either of the other pairs of opposite edges.

10. If P, Q are the middle points of the opposite edges BC, AD of a tetrahedron $ABCD$, prove that

$$AD^2+BC^2+4PQ^2=AC^2+BD^2+AB^2+CD^2$$

11. In a tetrahedron, if two pairs of opposite edges are at right angles, the same is true for the third pair, and the sum of the squares on pairs of opposite edges are equal.

[Let P, Q, R, S, T, U be the midpoints of AB, CD; AD, BC; AC, BD respectively. If the first two pairs of edges are at rt. angles, the figures $PTQU$, $STRU$ are rectangles. $\therefore PQ=TU=RS$. Hence $PQRS$ is also a rectangle. $\therefore$ etc.]

12. If opposite edges of a tetrahedron are mutually at right angles, prove that the normals from the vertices on the opposite faces are concurrent. [Such a tetrahedron is said to be *orthocentric*.]

13. Prove that the section of a pyramid by a plane parallel to the base is a polygon similar to the base.

14. Find the total area of a square pyramid whose base is a square of side $2h$, and whose height is h. Find the angle between an edge and the base.

15. The outer radius of a hollow right circular cylinder is 50 cm and the thickness is 2 cm. Find its weight, if 1 cu cm of the material weighs 7·6 g.

16. A cylinder is constructed by rolling up a rectangle of sides a and b about the edge b. If there is no over-lapping, find the radius of the base of the cylinder.

17. A right circular cone of height h and semi-vertical angle α is opened out into a plane. Find the angle of the circular sector obtained.

18. A circular sector of radius r and containing an angle θ is rolled into a right circular cone. Find the semi-vertical angle of the cone.

Imp **19.** A right circular cone is of height h and radius of base a. Find (i) the slant height, (ii) the area of the curved surface, (iii) the volume of the cone.

20. AB is a diameter of a sphere, and P any point on the sphere. Prove that $A\hat{P}B=90°$.

21. A and B are two fixed points. Find the locus of a point P lying in a plane p such that PA^2+PB^2 is constant.

22. Perpendiculars are drawn from a fixed point O on planes passing through a given line not passing through O. Prove that the locus of their feet is a sphere.

23. A fixed sphere is cut by any plane passing through a given point. Prove that the centre of the section lies on another sphere.

24. If a sphere can be placed in a wire tetrahedron so as to touch all the edges, prove that the sums of the pairs of opposite edges are equal.

25. A right circular cylinder, a right circular cone and a hemisphere have a common base and the same height. Prove that the volume of the cylinder is equal to the sum of the volumes of the cone and the hemisphere.

CHAPTER 5

Trigonometry; Radian Measure; Trigonometric Ratios

1. The word 'Trigonometry' suggests measurements of a triangle. A study of the relations that exist between the sides and angles of a triangle would be helpful in calculating some of these, when we are given the values of some others. The study of these and many other problems will be based on the values of certain trigonometrical ratios of angles. They are also called trigonometric functions. For every given angle, there will be a unique set of trigonometrical ratios of real numbers. Ultimately, then trigonometry will reduce itself to the algebra of these ratios or functions.

2. We begin with the notion of an angle. In our study of geometry, we introduced an angle as the set of all points on two distinct rays with a common end point. We propose to be a little more specific now by requiring the rays to be *ordered*, *i.e.*, by fixing one of them as the initial ray and the other as the terminal ray. The angle is said to be generated by the rotation of a ray about its end point from its initial position to its final position. The amount of rotation gives the measure of the angle. Thus the angle is a measure of the amount of rotation of a ray from one position to another. The angle is named as $\angle AOB$ or $A\hat{O}B$, which indicates the order; the middle letter O is the common end point of the two rays and is called the vertex of the angle. OA is the initial ray, OB the terminal ray.

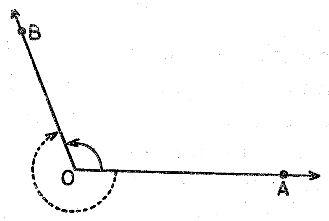


Fig. 5.1

The direction of rotation is very important. Thus, in Fig. 5.1 the bent arrow indicates a rotation from OA to OB in a direction opposite to the motion of the hands of a clock. This direction is called *counter—(or anti)* clockwise. But the other arrow shown by dots is a rotation from OA to OB in a *clockwise* direction. The angle $A\hat{O}B$ given in this manner is different from the angle previously described.

We shall now introduce a convention. If the rotation is counter-clockwise, the angle is said to be positive, otherwise it is negative.

We next see from Figs 5.2 and 5.3 that we can get different angles with the same vertex, initial and terminal rays. Thus in Fig. 5.2 a ray rotating from OA completes two revolutions round O and then rotates further to the position OB. The rotation is counter-clockwise. In Fig. 5.3 a ray starts from OA and rotates in the clockwise sense, completing one revolution and makes a further rotation to the position OB. In either case, the number of revolutions could be increased, making them as many times as we like.

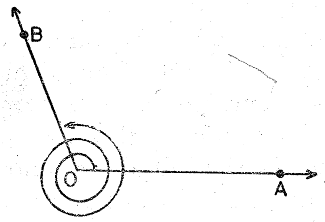


Fig. 5.2

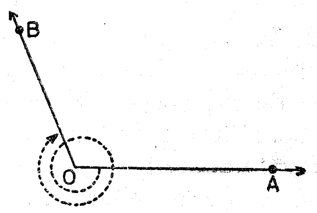


Fig. 5.3

When the initial and terminal rays are opposite to each other, the angle is called a straight angle and its measure is taken as 180° if the rotation is counter-clockwise. Half a straight angle gives for the terminal ray a position which is said to be *perpendicular* or at right angles to the initial ray and the measure of a right angle is, therefore, 90°. The symbol ° denotes degrees as given by a protractor. A complete revolution is equivalent to two straight angles and therefore, corresponds to an angle which is $+360°$ if the rotation is counter-clockwise and $-360°$ if it is clockwise. With any angle according to the amount of rotation involved and its direction, there corresponds a definite real number, which gives the number of degrees contained in the angle. Conversely to any given real number k, there corresponds one and only one angle having k degrees. If $k=0$, the angle is a zero angle, in which case OA and OB coincide.

It will often be convenient, especially in experimental work, to have sub-divisions of a degree, defined as follows:

one degree=60 minutes, 1 minute=60 seconds.

We use the symbol ′ for minute and ″ for second.

Example 1: The angle AOB marked by the arrow in Fig. 5.4 has the measure 45°.

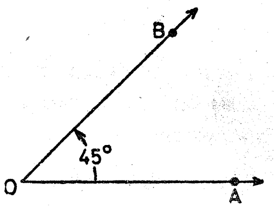


Fig. 5.4

(a) What other angles can also be denoted by the initial ray OA and final ray OB, the rotation being counter-clockwise?

The answer is $45°+360°$, $45°+720°$, $45°+(360\times3)°$ and so on.

(b) If the rotation is clockwise, what angles will be denoted by $A\hat{O}B$?

The answer is $-360°+45°$, *i.e.*, $-315°$, $-720°+45°$ and so on.

(c) If $A\hat{O}B=45°$, what is angle $B\hat{O}A$?

The answer is $-45°$, or $-45°-360°$, etc.

2. Using the protractor, construct angles whose values are (i) 60°, (ii) −60°, (iii) −210°, (iv) −780°, (v) 270°, (vi) −270°, (vii) 1080°, (viii) 1125°, (ix) −2250°.

2. We shall now develop a different mode of measuring an angle.

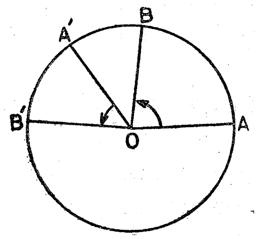


Fig. 5.5

We consider different angles $A\hat{O}B$, $A'\hat{O}B'$, etc. with the same vertex. We draw a circle with centre O and any radius r, cutting the rays of the angles at (A, B), (A', B') etc. Since $\angle AOB$ has its vertex at the centre of the circle, it is sometimes called a *central angle* of the circle.

Postulate. **The ratio of any two arcs of a circle is equal to the ratio of the angles subtended by them at the centre.**

Definition : The angle $A\hat{O}B$ is called the angle subtended by the arc AB (or by two points A and B) at the centre O of a circle.

If arc $AB=a$ and $A'B'=b$, we have by the postulate $\frac{a}{b}=\frac{A\hat{O}B}{A'\hat{O}B'}$.

Let us now choose an angle $A\hat{O}B$ such that the arc AB on the circle is equal to the radius. The measurement of the angle AOB is then called a *radian*. The expression of an angle in radians is called its circular or radian measure.

Definition : A radian is the measure of the angle subtended at the centre of a circle by an arc equal to the radius.

This gives us a new unit of measurement for measuring angles. If any other arc of length s on the same circle subtends an angle which may be denoted by θ (the Greek letter Theta) measured in radians, then according to the postulate,

$$\frac{s}{r}=\frac{\theta}{1} \quad \therefore\ s=r\theta \text{ or } \theta=\frac{s}{r}.$$

The number $\frac{s}{r}$ gives the value of the angle according to this new unit of measurement. According to this method, an angle is denoted by a number only, and not as so many degrees. When the angle is expressed by a number only, it is understood that it is in radian measure.

Let us now consider a straight angle. Hence, if we now compare a straight angle and an angle of one radian, whose corresponding arcs are respectively half the circumference of the circle, and a length equal to the radius, we have

$$\frac{\text{semi-circumference}}{r}=\frac{\text{straight angle}}{\text{one radian}}$$
$$=\text{constant by the postulate.}$$

In other words, whatever the radius of the circle, the ratio of the semi-circumference to the radius is a constant number. Or, what is the same thing, the **ratio of the circumference of any circle to its diameter is constant.** This constant ratio is denoted by the number π. Hence if r is the radius of the circle, its circumference $=2\pi r$.

The value of π cannot be expressed in the form of a rational number $\frac{a}{b}$, where a and b are integers. It is an irrational number. Rational numbers approximately equal to π can be given in various forms. Thus π is approximately equal to $\frac{22}{7}$, the value given by the Greek Mathematician Archimedes. A better approximation is 3·1416, a value given by the Indian mathematician Arya Bhata (499 A.D.). To ten decimal places, the value of π is 3·1415926536.

Since a straight angle is equal to 180°, we also have the result:

$$\frac{180^\circ}{1 \text{ radian}}=\frac{\pi}{1}, \text{ or } \pi \textbf{ radians}=180^\circ$$

This gives the conversion formula from degrees to radians or vice versa.

For instance, $90^\circ=\frac{\pi}{2}$; $60^\circ=\frac{\pi}{3}$; $30^\circ=\frac{\pi}{6}$; $120^\circ=\frac{2\pi}{3}$ in radians.

Exercise 1

1. Find the radian measure of the following angles.

(i) 270°, (ii) 315°, (iii) −840°, (iv) $\frac{1}{2}°$, (v) −3645°, (vi) 1″.

2. Prove that one radian is equal to 57° 17′ 45″ approximately. [Assume $\pi=\frac{22}{7}$.]

3. Find the number of degrees (approximately, if the value cannot be found exactly) in the angles whose radian measures are

(i) $\frac{\pi}{12}$, (ii) $\frac{7\pi}{5}$, (iii) $-\frac{17\pi}{2}$, (iv) 2, (v) 5, (vi) $\frac{\pi}{7}$.

4. Find in radian measure, a vertical angle of

(i) an equilateral triangle, (ii) a regular pentagon, (iii) a regular heptagon.

5. An angle whose radian measure is 0·75 is subtended by an arc 30 metres in length. Find the radius of the circle.

6. The minute hand of a clock is 40 cm long. Through how many centimetres does its extremity move in 5 minutes? [Take $\pi=\frac{22}{7}$.]

7. The fly wheel of an engine makes 50 revolutions per second. How long will it take to turn through (i) a right angle, (ii) 5 radians?

8. Find the length of an arc on the surface of the earth which subtends 1 minute at its centre. Earth's radius =6480 km.

9. Two places on the same meridian of the earth are 300 km apart. Find their difference in latitude. –Earth's Rdu = 6480 km.

10. A wire of length 10 cm is bent into the arc of a circle of radius 4 cm. What is the angle formed at the centre?

11. Taking the earth's radius as 6400 km, find to the nearest km the distance between two places on the same meridian of the earth whose latitudes differ by one degree.

12. A pendulum 7 m long oscillates through an angle of 10°. What is the length of the path its extremity describes between its extreme positions?

13. A wheel, radius 50 cm is spinning on its axis at 3 radians per second. Find the speed of a point on its rim.

3. Trigonometrical Functions. We shall first consider an acute angle AOB.

From any point P on the terminal ray, draw PQ perpendicular to the initial ray OA. Let $\angle AOB$ be called θ. We define, in terms of the sides of the right-angled triangle POQ,

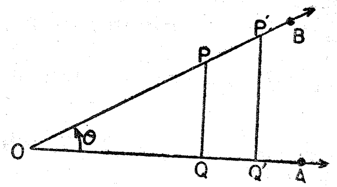


Fig. 5.6

$$\text{sine } \theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{PQ}{OP}$$

$$\text{cosine } \theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{OQ}{OP}$$

$$\text{tangent } \theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{PQ}{OQ}$$

The reciprocals of these expressions are respectively called cosecant θ, secant θ, and cotangent θ. These six ratios are called the trigonometrical functions or ratios of the angle θ. They will be written hereafter abbreviated as sin θ, cos θ, tan θ, cosec θ, sec θ, cot θ respectively.

We first of all note that these trigonometrical ratios do not depend on the choice of the point P on the ray OB. For, if any other point P' be taken on the ray and $P'Q'$, be drawn perpendicular to OA, the two triangles OPQ and $OP'Q'$ are similar.

$$\therefore \frac{PQ}{OP}=\frac{P'Q'}{OP'};\ \frac{OQ}{OP}=\frac{OQ'}{OP'},\text{ and so on.}$$

The ratios sin θ, cos θ and tan θ are called the *primary trigonomertical ratios*; for, the other three ratios cosec θ, sec θ and cot θ are merely the reciprocals of sin θ, cos θ and tan θ.

If we take OA as the x-axis, the perpendicular to OA as the y-axis and the coordinates of a point P on OB, as (x, y), then $\sin\theta=\frac{y}{r}$, $\cos\theta=\frac{x}{r}$ and $\tan\theta=\frac{y}{x}$, where $r=OP=\sqrt{x^2+y^2}$.

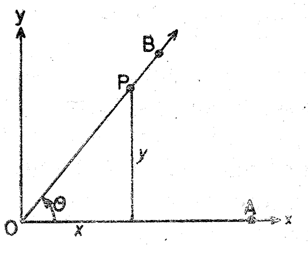


Fig. 5.7

Here θ is an acute angle; but these trigonometrical functions can be defined for any angle. If OB is in the second quadrant, the x-coordinate of P is negative and the y-coordinate is positive. Here θ lies betwen $\frac{\pi}{2}$ and π.

$$\therefore \sin\theta=\frac{y}{r}\text{ is positive and }\cos\theta=\frac{x}{r}\text{ is negative.}$$

Similarly if θ lies between π and $\frac{3\pi}{2}$, the terminal ray OB is in the third quadrant and both x and y, the coordinates of P are negative.

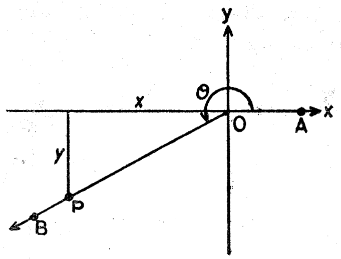


Fig. 5.8

$\therefore \sin\theta=\frac{y}{r}$ and $\cos\theta=\frac{x}{r}$ are both negative; but $\tan\theta=\frac{x}{y}$ is positive.

Lastly if the terminal ray is in the fourth quadrant, x is positive and y is negative.

$\therefore$ sin θ is negative and cos θ is positive.

Here θ lies between $\frac{3\pi}{2}$ and 2π.

We have also seen that an angle may be measured in the positive sense or in the negative sense. Even if θ is negative, the coordinates of P can be found. In this figure, θ is negative and lies in the fourth quadrant. The x-coordinate of P is positive and the y-coordinate is negative. Sin $\theta=\frac{y}{r}$ is negative and $\cos\theta=\frac{y}{r}$ is positive. We notice that if θ is positive,

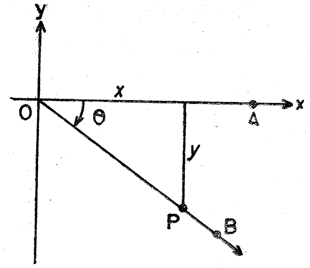


Fig. 5.9

$$\sin(-\theta)=-\sin\theta\text{ and }\cos(-\theta)=\cos\theta.$$

This is true whatever be the value of θ.

Exercise. Calculate the trigonometrical ratios of each of the following angles:

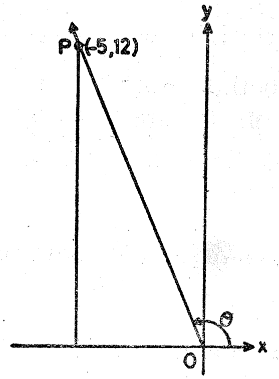


Fig. 5.10

In the first figure, the terminal ray of θ lies in the second quadrant *i.e.*, lies between $\frac{\pi}{2}$ and π, $x=-5, y=12$, and

$$r=\sqrt{x^2+y^2}=\sqrt{25+144}=\sqrt{169}=13$$

$\therefore \sin\theta=\frac{12}{13}$, $\cos\theta=-\frac{5}{13}$ and $\tan\theta=-\frac{12}{5}$;

$\operatorname{cosec}\theta=\frac{13}{12}$, $\sec\theta=-\frac{13}{5}$ and $\cot\theta=-\frac{5}{12}$.

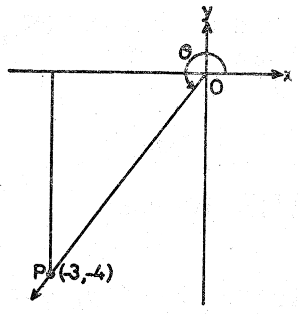


Fig. 5.11

In the second figure, $x=-3, y=-4$, and $r=5$. The terminal ray of θ lies in the third quadrant.

$\therefore \sin\theta=-\frac{4}{5}$, $\cos\theta=-\frac{3}{5}$ and $\tan\theta=\frac{4}{3}$

$\operatorname{cosec}\theta=-\frac{5}{4}$, $\sec\theta=-\frac{5}{3}$ and $\cot\theta=\frac{3}{4}$.

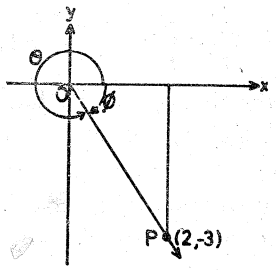


Fig. 5.12

In the third figure, the terminal ray of the angle θ lies in the fourth quadrant.

The coordinates of P are $(2, -3)$.

$x=2, y=-3$, and $r=\sqrt{4+9}=\sqrt{13}$

$\therefore \sin\theta=-\dfrac{3}{\sqrt{13}}$, $\cos\theta=\dfrac{2}{\sqrt{13}}$ and $\tan\theta=-\frac{3}{2}$

$\operatorname{cosec}\theta=-\dfrac{\sqrt{13}}{3}$, $\sec\theta=\dfrac{\sqrt{13}}{2}$ and $\cot\theta=-\frac{2}{3}$

Here the angle θ and angle ϕ (*i.e.* $P\hat{O}X$) have the same trigonometrical ratios. The angle ϕ is measured in the clockwise direction, while θ is measured in the anti-clockwise direction.

Sin $\phi=-\dfrac{3}{\sqrt{13}}$ and $\cos\phi=\dfrac{2}{\sqrt{13}}$. ϕ has a negative measure.

4. Identities.

1. We notice that $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$; for,

$$\sin\theta=\frac{y}{r} \text{ and } \cos\theta=\frac{x}{r} \text{ and } \tan\theta=\frac{y}{x}=\frac{y}{r}\div\frac{x}{r}=\frac{\sin\theta}{\cos\theta},$$

where $r=\sqrt{x^2+y^2}$.

2. $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{x}{y} = \frac{1}{\tan\theta}$.

3. We write $(\sin\theta)^2$ or $\sin\theta \times \sin\theta$ as $\sin^2\theta$ and $(\cos\theta)^2$ as $\cos^2\theta$.

Now $\sin^2\theta + \cos^2\theta = \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{x^2+y^2}{r^2} = 1$.

This follows from the theorem of Pythagoras. Dividing each side by $\cos^2\theta$ and $\sin^2\theta$, we have $\tan^2\theta + 1 = \sec^2\theta$ and $1+\cot^2\theta = \text{cosec}^2\,\theta$.

The main identity to remember is $\sin^2\theta+\cos^2\theta=1$. This may be written also as $1-\cos^2\theta=\sin^2\theta$ or as $1-\sin^2\theta=\cos^2\theta$.

Exercise 2

1. In each of the following cases, the coordinates of a point P on the terminal ray are given. Find the primary trigonometrical ratios of the angle AOP where A is a point on the x-axis. Draw the diagram in each case.

(1) $(-2, 3)$. (2) $(5, 4)$.
(3) $(3, -4)$. (4) $(-6, -8)$.
(5) $(7, -24)$. (6) $(-5, 12)$.

Write an equivalent to each of the following:

2. (i) $1-\sin^2\theta$, (ii) $1+\tan^2\theta$, (iii) $\cot\theta.\tan\theta$,

(iv) $\tan\theta\cdot\cos\theta$, (v) $\cot\theta\cdot\sin\theta$, (vi) $\frac{1}{\text{cosec}\,\theta}$,

(vii) $\frac{1}{\tan\theta}$.

3. If $\sin\theta=+\frac{3}{5}$ and $\cos\theta=-\frac{4}{5}$, draw a diagram and find the other four trigonometrical ratios of θ.

4. If $\sin\theta=-\frac{2}{5}$, find $\cos\theta$, $\tan\theta$ and $\cot\theta$. How many values can there be for these functions?

5. If $\cos\theta=-\frac{3}{8}$, in which quadrant should the terminal ray of θ lie? What is $\cos(-\theta)$?

6. If $\tan\theta=\frac{2}{3}$, in which quadrant should the terminal ray of angle θ lie? What is $\tan(-\theta)$?

7. The angle θ lies between π and $\frac{3\pi}{2}$. What are the signs of $\sin\theta$, $\cos\theta$ and $\tan\theta$?

8. The angle θ lies between $\frac{3\pi}{2}$ and 2π. What are the signs of $\sin\theta$, $\cos\theta$ and $\tan\theta$?

Prove the following identities:

9. $\sin^2 A-\cos^2 A=1-2\cos^2 A=2\sin^2 A-1$.

10. $\tan B+\cot B=\frac{1}{\sin B\cos B}$.

11. $\tan\alpha+\cot\alpha=\sec\alpha\cdot\text{cosec}\,\alpha$.

12. $(1-\sin^2\alpha)\sec^2\alpha=1$.

13. $\frac{\sin^2\alpha}{1-\cos^2\alpha}=1$.

14. $\sin^2\beta\cot^2\beta\sec^2\beta=1$.

15. $\tan^2\alpha(\text{cosec}^2\alpha-1)=1$.

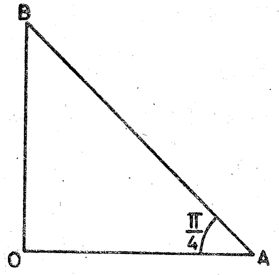


Fig. 5.13

4. The trigonometrical ratios of $\frac{\pi}{4}$, $\frac{\pi}{6}$, $\frac{\pi}{3}$.

(a) Consider a right-angled triangle AOB with the right angle at O.

Let the angle A be 45° or $\frac{\pi}{4}$. Angle B in that case is also $=45°$ or $\frac{\pi}{4}$.

If $OA=a$, $OB=a$ and $AB=\sqrt{a^2+a^2}$
$=a\sqrt{2}$

Hence considering $\angle OAB$

$$\sin 45° = \frac{OB}{AB} = \frac{1}{\sqrt{2}};\ \cos 45° = \frac{OA}{AB} = \frac{1}{\sqrt{2}};\ \tan 45° = \frac{OB}{OA} = 1.$$

In radian measure,

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}},\ \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}};\ \tan\frac{\pi}{4} = 1.$$

(b) Next, consider triangle AOB, in which $\hat{O}=90°$, $O\hat{A}B=30°$ $\hat{O}BA=60°$.

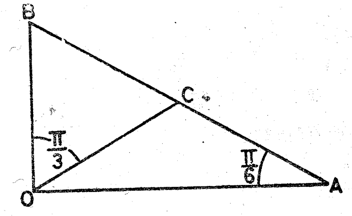


Fig. 5.14

At O make $B\hat{O}C=60°$. Hence BOC is equilateral,

$\therefore\ OB=BC=OC=a$ (say)

Also $C\hat{A}O=C\hat{O}A=30°$ $\quad\therefore\ CA=CO=a$

$\therefore\ AB=2a$ and $OA=\sqrt{(2a)^2-a^2}=a\sqrt{3}$

$\therefore$ considering $O\hat{A}B$,

$$\sin 30° = \frac{OB}{AB} = \tfrac{1}{2};\ \cos 30° = \frac{OA}{AB} = \frac{\sqrt{3}}{2};\ \tan 30° = \frac{AB}{OA} = \frac{1}{\sqrt{3}}$$

Considering $A\hat{B}O$

$$\sin 60° = \frac{OA}{AB} = \frac{\sqrt{3}}{2};\ \cos 60° = \frac{OB}{AB} = \tfrac{1}{2};\ \tan 60° = \frac{OA}{OB} = \sqrt{3}$$

In radian measure

$$\sin\frac{\pi}{6} = \tfrac{1}{2},\ \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2},\ \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}};$$

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2},\ \cos\frac{\pi}{3} = \tfrac{1}{2},\ \tan\frac{\pi}{3} = \sqrt{3}.$$

$$\sin^2\frac{\pi}{6} + \cos^2\frac{\pi}{6} = \tfrac{1}{4} + \tfrac{3}{4} = 1 \text{ and}$$

$$\sin^2\frac{\pi}{3} + \cos^2\frac{\pi}{3} = \tfrac{3}{4} + \tfrac{1}{4} = 1.$$

Also $\sin^2\frac{\pi}{4} + \cos^2\frac{\pi}{4} = \frac{1}{2} + \frac{1}{2} = 1.$

Quadrantal angles: When the initial ray coincides with the positive part of the x-axis and the terminal ray coincides with the positive part of the (i) y-axis or (ii) the negative part of the x-axis, or (iii) the negative part of the y-axis, we have angles whose measures are $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$. We can also imagine that the terminal ray coincides with Ox itself, *i.e.*, with the positive part of the x-axis. In that case the angle formed is zero. These are called *quadrantal* angles. Take P with coordinates (1, 0), (0, 1), (−1, 0) and (0, −1) in the four positions.

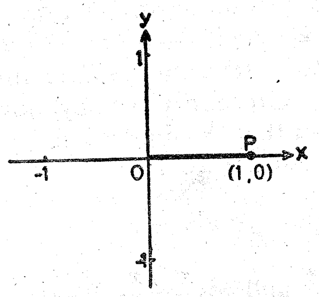


Fig. 5.15 (a)

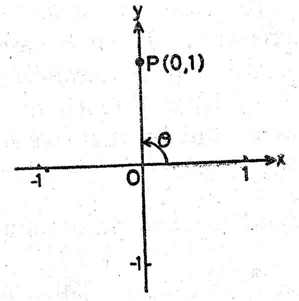


Fig. 5.15 (b)

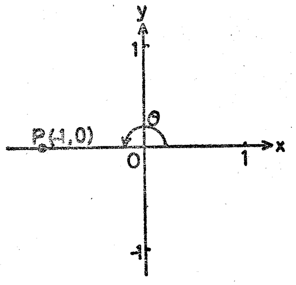


Fig. 5.15 (c)

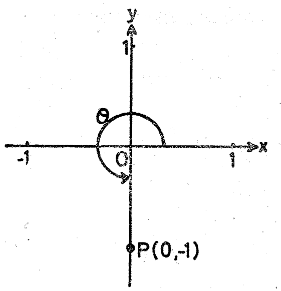


Fig. 5.15 (d)

Trigonometrical ratios for angle 0°.

$$\sin 0^\circ = \frac{y}{r} = \frac{0}{1} = 0; \cos 0^\circ = \frac{x}{r} = \frac{1}{1} = 1,$$

as $x=1, y=0$ and $r=1$ in this case.

$\tan 0^\circ = \frac{0}{1} = 0$, but cot 0°, cosec 0° are not defined in this case, as division by zero is not permissible. When θ is a very small number, $\cot \theta$ is a big number. For instance, if in radian measure θ is $\frac{1}{100}$, the arc of the circle of unit radius subtending this angle is also $\frac{1}{100}$.

$\therefore y$ is a number slightly less than $\frac{1}{100}$.

$$\therefore \cot \theta = \frac{x}{y} > 100.$$

If $\theta = \frac{1}{1000}$, $\cot \theta > 1000$. Thus the smaller the angle θ, the greater the value of $\cot \theta$. If θ goes on decreasing, $\cot \theta$ goes on increasing. In such cases, we say if $\theta \to 0$, $\cot \theta \to \infty$ (infinity), the sign $\to$ standing for 'tends to'. The limit of $\cot \theta$ is ∞ when θ tends to 0. Ordinarily we say, however, $\cot 0 = \infty$; but by that, we mean that the limit is infinity when the angle tends to 0. Similarly $\operatorname{cosec} 0 = \infty$.

Trigonometrical ratios for angles $\frac{\pi}{2}$, and $\pi\,\frac{3\pi}{2}$.

$$\sin \frac{\pi}{2} = \frac{y}{r} = \frac{1}{1} = 1, \text{ when } \theta = \frac{\pi}{2} \text{ and } \cos \frac{\pi}{2} = \frac{x}{r} = \frac{0}{1} = 0.$$

$$\therefore \sin \frac{\pi}{2} = 1, \cos \frac{\pi}{2} = 0, \tan \frac{\pi}{2} = \infty \text{ and so on.}$$

Also $\sin \pi = 0$, $\cos \pi = -1$, $\tan \pi = 0$ and $\sin \frac{3\pi}{2} = -1$,

$$\cos \frac{3\pi}{2} = 0 \text{ and } \tan \frac{3\pi}{2} = \infty.$$

Exercise 3

1. Find the angle θ where $0 \leqslant \theta^\circ \leqslant 90^\circ$ and (a) $\sin \theta = \frac{1}{2}$, (b) $\cos \theta = \frac{1}{\sqrt{2}}$, (c) $\tan \theta = 1$.

2. Find the angle θ where $\frac{\pi}{2} \leqslant \theta \leqslant \pi$ and

(i) $\sin \theta = \frac{\sqrt{3}}{2}$, (ii) $\tan \theta = -1$, (iii) $\tan \theta = 0$.

3. Find the angle θ where $\pi \leqslant \theta \leqslant \frac{3\pi}{2}$ and

(i) $\cos \theta = 0$, (ii) $\tan \theta = \sqrt{3}$, (iii) $\sin \theta = -\frac{\sqrt{3}}{2}$.

4. Find the angle θ where $\frac{3\pi}{2} \leqslant \theta \leqslant 2\pi$ and

(i) $\cos \theta = \frac{1}{\sqrt{2}}$, (ii) $\tan \theta = -1$, and (iii) $\sec \theta = 1$.

5. Find the value of $\sin^2 \frac{\pi}{3} + \cos^2 \frac{\pi}{6}$.

6. Find the value of $\cos^2 30^\circ - \cos^2 45^\circ$.

7. Determine the value of $2 \sin 30^\circ \cos 30^\circ \cot 60^\circ$.

8. Determine the value of $3 \cos \frac{\pi}{2} - 2 \cos \pi$.

9. Evaluate $\tan^2 \frac{\pi}{6} - 2 \sec^2 \frac{\pi}{4} + \cot^2 \frac{\pi}{6}$.

10. Evaluate $\cos^2 0° \sin^2 45° - \cot 90° \sin 45°$.

11. Prove that $2 \left(\sin^2 \frac{\pi}{6} + \sin^2 \frac{\pi}{4} + \sin^2 \frac{\pi}{3} \right) = 3$.

12. Prove that $\sin \frac{\pi}{3} \cos \frac{\pi}{6} + \cos \frac{\pi}{3} \sin \frac{\pi}{6} = 1$.

13. Prove that $\tan^2 30° + 2 \sec^2 45° + \cot^2 60° = 4\frac{2}{3}$.

14. Determine the value of $2 \tan^2 \frac{\pi}{4} + \tan^2 \frac{\pi}{3}$.

15. Determine $\sec^2 45° + \operatorname{cosec}^2 45° - 2 \sin 45° \cos 45°$.

PART II

COORDINATE GEOMETRY

CHAPTER 6

Distance and Points of Division of a Line Segment.

1. Introduction.

Coordinate Geometry is also known as Analytical Geometry. It links Algebra and Geometry. With its aid, we can solve geometrical problems by using algebraic methods.

In earlier courses in Algebra, you have made use of coordinate axes while drawing simple graphs. By selecting coordinate axes and a unit of length, it is possible to assign to each point in a plane an ordered pair of real numbers (x, y). We divide the plane into four quadrants by two perpendicular lines, intersecting at a point O, known as the origin.

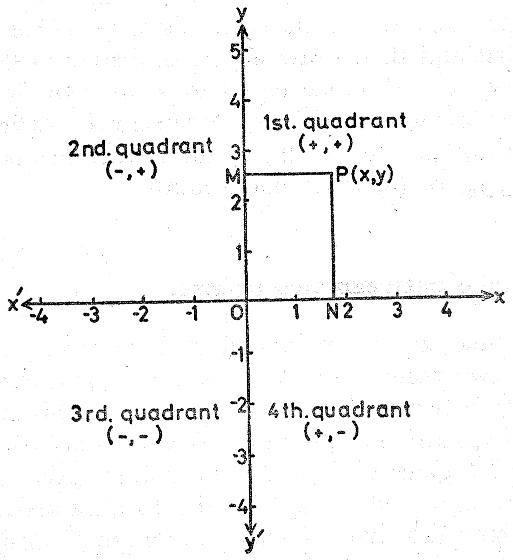


Fig. 6.1

The horizontal line XOX' is called the x-axis and the vertical line YOY' is called the y-axis. The distance PM from the y-axis of any point P is called the x-coordinate or the *abscissa*, and the distance PN from the x-axis is called the y-coordinate or the *ordinate*. In the figure above, $PM=ON$ and $PN=MO$. In the first quadrant, the coordinates x, y of any point P are both positive, in the second quadrant, x is negative and y is positive, in the third quadrant both x and y are negative and in the fourth quadrant x is positive and y is negative. Abscissae are positive when measured to the right of the y-axis and negative when measured to the left. Ordinates are positive when measured above the x-axis, negative if measured below.

If the coordinates x, y of a point are given, the point can be located on the plane. Then the point is said to be plotted. For this, a suitable scale has to be chosen first, say 1 cm. = a unit and the distances x and y are to be marked on the two axes. In actual practice, we measure the distance ON on the x-axis $=x$ units first and then draw a perpendicular to the x-axis at N and measure a distance equal to y units on it. x and y are said to form an ordered pair, because x comes first and then y. It is customary to use (x, y) for a variable point P and (x_1, y_1), (x_2, y_2) or (a, b) for fixed points.

2. Distance between two points.

(i) First take any two points which have the same ordinate. Let A, B be two points $(-3, 2)$ and $(1, 2)$. You can easily see the distance between A and B as 4 units. This is really the difference between the abscissae of two points $(-3, -1)$ or $1-(-3)$. In general, if the two points have coordinates (x_1, y_1) and (x_2, y_2) and $y_1=y_2$, then the distance between them is x_2-x_1 ignoring the sign. Hence the length of a line segment parallel to the x-axis and with end points $A(x_1, y_1)$ and $B(x_2, y_1)$ is $|x_2-x_1|$ *i.e.*, x_2-x_1 if $x_2>x_1$ and x_1-x_2 if $x_1>x_2$.

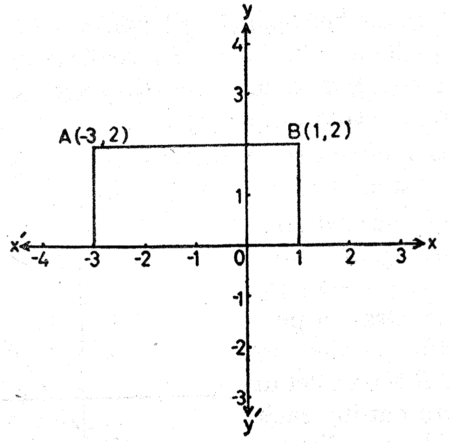


Fig. 6.2

(ii) Now take any two points which have the same, abscissae for example $A(2, 3)$, $B(2, -5)$.

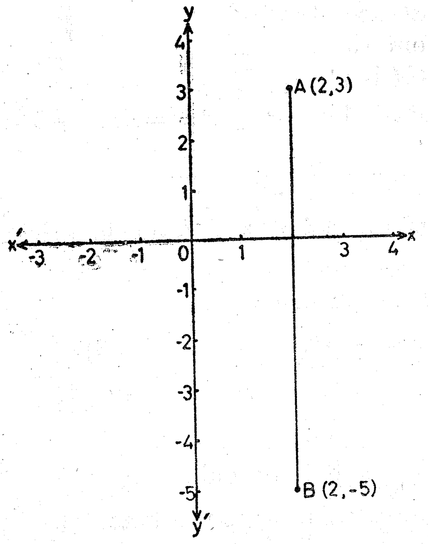


Fig. 6.3

Here the distance between A and B is clearly 8 units. This is the same as $3-(-5)$ units. In general if $A(x_1, y_1)$ and $B(x_1, y_2)$ are any two points, the distance between them is y_2-y_1 ignoring the sign or $|y_2-y_1|$.

Length of a line segment parallel to the y-axis and with end points $A(x_1, y_1)$ and $B(x_1, y_2)$ is $|y_2-y_1|$.

(iii) The distance between any two points (Fig. 6.4). Now take any two points $A(x_1, y_1)$ and $B(x_2, y_2)$. Draw a perpendicular AM parallel to the x-axis and BM parallel to the y-axis, both cutting each other at M. The coordinates of M are (x_2, y_1), as A and M are at the same distance from the x-axis. From (i) above, the distance AM is x_2-x_1 and from (ii) above, the distance BM is y_2-y_1.

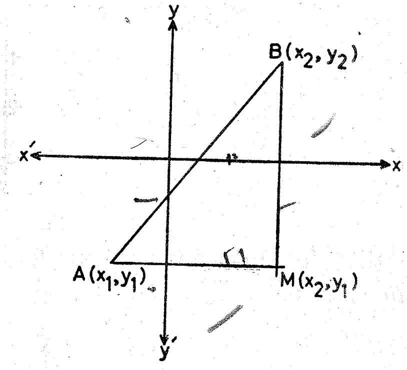


Fig. 6.4

By Pythagoras' Theorem, $AB^2=AM^2+BM^2=(x_2-x_1)^2+(y_2-y_1)^2$.

$$\therefore\ AB=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}.$$

The distance of any point (x, y) from the origin is $\sqrt{x^2+y^2}$.

Example 1: Find the distance of the line segment between the points $A(2, -3)$ and $B(-5, 2)$.

$$\begin{aligned}\text{The distance} &= \sqrt{(-5-2)^2+(2+3)^2}\\ &= \sqrt{7^2+5^2} \quad =\sqrt{49+25}\\ &= \sqrt{74}\\ &= 8{\cdot}6 \text{ approximately.}\end{aligned}$$

Note that while calculating the distance between two points, we calculate the numerical length ignoring the negative sign of the square root. It is understood that all the distances are in suitable units.

Example 2: Show that $A(-1, -5)$, $B(1, -1)$, $C(2, 1)$ are collinear, *i.e.*, are in the same straight line.

Here we shall show that $AB+BC=AC$.

$$\begin{aligned}AB &= \sqrt{(-1-1)^2+(-5+1)^2}=\sqrt{(4+16)}=\sqrt{20}=2\sqrt{5}.\\ BC &= \sqrt{(2-1)^2+(1+1)^2}=\sqrt{1+4}=\sqrt{5}\\ AC &= \sqrt{(2+1)^2+(1+5)^2}=\sqrt{9+36}=\sqrt{45}=3\sqrt{5}\end{aligned}$$

$\therefore$ $AB+BC=2\sqrt{5}+\sqrt{5}=3\sqrt{5}=AC$, taking only the positive signs of the square roots.

$\therefore$ A, B, C are collinear.

Exercise 1

Determine the lengths of the line segments joining the origin to the points whose coordinates are

1. (3, 4). **2.** (5, 12). **3.** (7, 24). **4.** (3, 5).

5. $\left(\frac{a}{2}, \frac{b}{2}\right)$. **6.** $(a+b, a-b)$. **7.** $\left(\frac{1}{a}, \frac{1}{b}\right)$.

8. Find the value of x so that the length of the segment joining the origin to $P(6, x)$ is 10.

Calculate the lengths of the segments whose end points are:

9. (0, 0), (−4, 3).
10. (−5, 0), (−2, 0).
11. (2, −1), (−1, −1).
12. $(a, 3)$, $(a, -5)$.
13. $(a, -b)$, (a, b).
14. (−3, 5), (1, 2).
15. (−3, 2), (3, 10).
16. (−2, −5), (−7, 0).
x_1 y_1 x_2 y_2
17. (−1, 8), (3, 5).
18. (7, 13), (−2, 1). ✓

19. Plot the points (−2, 0), (0, 3), (2, 0) and show that the line segments joining the points form an isosceles triangle.

20. Find the perimeter of the triangle PQR, determined by $P(1, 3)$, $Q(4, 7)$, $R(1, 11)$.

21. Show that the following points are the vertices of an isosceles triangle.

(i) $A(2, 1), B(3, 4), C(4, 1)$

(ii) $P(-8, -2), Q(-11, 3), R(3, 8)$.

22. Show that the points $A(-1, 0)$, $B(1, 0)$, $C(0, \sqrt{3})$ form an equilateral triangle.

23. Show that the points $A(-3, -4)$, $B(1, -1)$ and $C(5, 2)$ lie in a straight line.

24. Show that the triangle with the vertices $P(-4, 2)$, $Q(-4, 6)$ and $R(7, 2)$ is a right-angled triangle.

25. Show that the point (2, 3) is the midpoint of the line segment whose end points are (1, 1) and (3, 5).

26. Find the value of x so that $(x, 0)$ is equidistant from $P(3, 2)$, $Q(5, 3)$.

27. Determine the point which is equidistant from the points $A(-3, 3)$, $B(4, 4)$, $C(6, 0)$. What is this point called?

28. Show that (4, 3) is the centre of the circle circumscribing the triangle with vertices $A(1, 7)$, $B(7, -1)$, $C(8, 6)$.

29. Determine the coordinates of the centre of the circle on which the points $A(6, 0)$ $B(1, 5)$, $C(5, 3)$ lie.

30. Show that the points $(a, 0)$, $(2a, -b)$ and $(-a, 2b)$ are collinear.

3. Point of division of a directed line segment.

(a) *Internal division.* If C is a point on a directed line AB, where A is the initial point and B is the terminal point, then AB is said to be divided internally at C in the ratio $AC : CB$ or $m : n$, where m and n are positive real numbers. We agree to call the direction from the initial to the terminal point as positive and that the direction from the terminal point to the initial line as negative. If $AC : CB = m : n$, then $BC : CA = n : m$.

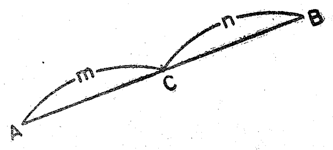


Fig. 6.5

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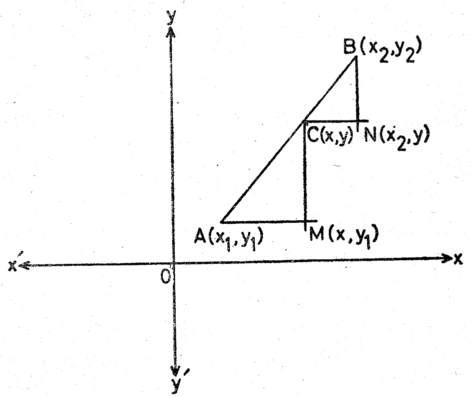

Fig. 6.6

Let the coordinates of A, B and C be respectively (x_1, y_1), (x_2, y_2) and (x, y). Draw line segments AM and CM parallel to the axes meeting at M. Similarly draw line segments CN and BN parallel to the axes meeting at N. The coordinates of M and N are (x, y_1) and (x_2, y) respectively. The two triangles ACM and BCN are similar.

$$\therefore \frac{AM}{CN} = \frac{x-x_1}{x_2-x} = \frac{AC}{CB} = \frac{m}{n}$$

$$\frac{CM}{BN} = \frac{y-y_1}{y_2-y} = \frac{AC}{CB} = \frac{m}{n}$$

$$\therefore n(x-x_1) = m(x_2-x)$$

$$\therefore (m+n)x = mx_2+nx_1$$

or $x = \dfrac{mx_2+nx_1}{m+n}$. Similarly $y = \dfrac{my_2+ny_1}{m+n}$.

$\therefore$ The coordinates of C are $\dfrac{mx_2+nx_1}{m+n}, \dfrac{my_2+ny_1}{m+n}$.

Thus the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ is divided internally in the ratio $m : n$ at a point

$$C\left(\frac{mx_2+nx_1}{m+n},\ \frac{my_2+ny_1}{m+n}\right).$$

As a particular case, the coordinates of the *midpoint* of the line segment AB are $\left(\frac{x_1+x_2}{2},\ \frac{y_1+y_2}{2}\right)$; for, $m=n$ in this case.

(b) External division of a directed line.

If C is a point on the extension of the directed line segment AB, then AB is divided externally at C in the ratio $AC : CB$ or $m : -n$, where m and n are positive real numbers. Here AC is in the positive direction and CB is in the negative direction.

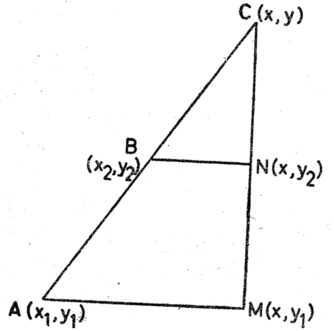

Fig. 6.7

Draw AM, CM parallel to the axes meeting at M. Draw BN parallel to the x-axis meeting CM at N. The coordinates of M and N are (x, y_1) and (x, y_2).

From the properties of similar triangles ACM and BCN, we have

$$\frac{CB}{CA}=\frac{BN}{AM}=\frac{CN}{CM}=\frac{n}{m} \text{ (as } CB, CA \text{ are in the same direction).}$$

Now $AM=x-x_1$ and $BN=x-x_2$

and $CM=y-y_1$ and $CN=y-y_2$.

$$\therefore \frac{x-x_2}{x-x_1}=\frac{n}{m} \text{ or } m(x-x_2)=n(x-x_1)$$

$$\therefore x(m-n)=mx_2-nx_1 \text{ or } x=\frac{mx_2-nx_1}{m-n}.$$

Similarly $y=\frac{my_2-ny_1}{m-n}$.

Hence the line segment joining two points (x_1, y_1) and (x_2, y_2) is divided in the ratio $m : -n$ at the point whose coordinates are

$$\frac{mx_2-nx_1}{m-n},\ \frac{my_2-ny_1}{m-n}.$$

A line segment is said to be divided *harmonically* when it is divided internally and externally in the same ratio. If the line segment whose end points are $A(x_1, y_1)$ and $B(x_2, y_2)$ is divided at C and D in the ratio of $m : n$ and $m : -n$ respectively, AB is divided harmonically at C and D. The coordinates of C are $\left(\frac{mx_2+nx_1}{m+n},\ \frac{my_2+ny_1}{m+n}\right)$ and the coordinates of D are $\left(\frac{mx_2-nx_1}{m-n},\ \frac{my_2-ny_1}{m-n}\right)$.

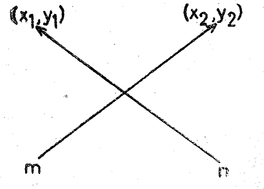

Fig. 6.8

The following array showing the coordinates of A and B and the ratio $m : n$ may help calculate the coordinates of the points of division in the given ratio.

Example 1: Find the coordinates of the point which divides the line segment whose end points are $A(1, 7)$, $B(6, -3)$ in the ratio $2 : 3$.

The point of internal division may be called C.

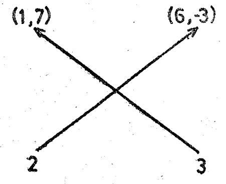

Fig. 6.9 (a)

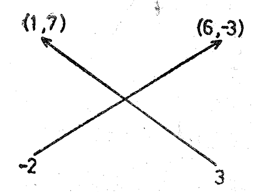

Fig. 6.9 (b)

The x-coordinate of C is $\dfrac{3+12}{5}=3$ and

the y-coordinate of C is $\dfrac{21-6}{5}=3$.

$\therefore$ the coordinates of C are (3, 3).

If D is the point of external division, the ratio is $-2:3$.

The x-coordinates of D is $\dfrac{3-12}{1}=-9$ and

the y-coordinate is $\dfrac{21+6}{1}=27$.

Example 2: A circle with centre $O(2, 6)$ has one end of a diameter at $A(3, 7)$. Determine the coordinates of the other end.

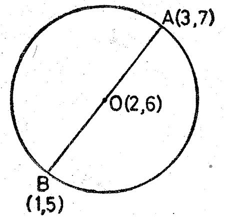

Fig. 6.10

Since O is the midpoint of the two ends of the diameter, $\dfrac{OA}{AB}=\dfrac{1}{-2}$

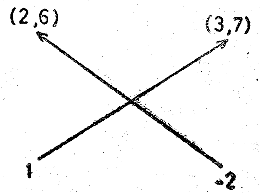

Fig. 6.11

The coordinates of B are found from the array.

$$x=\frac{-4+3}{-2+1}=1$$

$$y=\frac{-12+7}{-2+1}=5$$

$\therefore$ the coordinates of B are (1, 5).

Here O is the midpoint of BA.

$\therefore$ the coordinates of O are $\left(\dfrac{1+3}{2}, \dfrac{5+7}{2}\right)$ or (2, 6).

Exercise 2

Find the coordinates of the point dividing the line segment AB in the given ratios in the following cases

1. $A(a, b)$, $B(-a, -b)$; 1 : 1.
2. $A(2, -1)$, $B(3, 5)$; 1 : 2.
3. $A(5, 3)$, $B(2, 1)$; 2 : 1.
4. $A(3, -2)$, $B(5, 0)$; 2 : 3.
5. $A(1, 2)$, $B(5, 8)$; 3 : 5.
6. $A(0, 1)$, $B(3, -2)$; 2 : 3.
7. $A(-2, 3)$, $B(3, -2)$; 3 : −1.
8. $A(-3, -5)$, $B(1, 1)$; 1 : −3.
9. $A(-1, -2)$, $B(1, 0)$; −2 : 5.
10. $A(-1, 4)$, $B(2, -1)$; −1 : 4.
11. $A(5, -3)$, $B(7, 1)$; 2 : −3.
12. $A(0, 0)$, $B(2, -3)$; 5 : −1.

State the initial and terminal points of the line segment and the ratio in which it is divided at the point with the coordinates

13. $\left(\dfrac{2a+c}{3}, \dfrac{2b+d}{3}\right)$.
14. $\left(\dfrac{a-2c}{-1}, \dfrac{b-2d}{-1}\right)$.
15. $\left(\dfrac{3m-4n}{m-n}, \dfrac{2m-3n}{m-n}\right)$.
16. $\left(\dfrac{a+c}{2}, \dfrac{b+d}{2}\right)$.
17. Determine the two points of trisection of the line segment joining $A(3, -1)$ and $B(9, 7)$.
18. Find the coordinates of the three points dividing the line segment joining (−4, −8), and (2, 4) into four equal parts.
19. The point (−4, 1) divides the line segment with the initial point (2, −2) and terminating point (x, y) in the ratio 3 : 5. Find the values of x and y.
20. *Show that the abscissae of the centroid of a triangle is the average of the abscissae of the vertices and its ordinate is the average of the ordinates.*

CHAPTER 7

Straight Lines

While drawing graphs, you have already noticed that an equation like $3x-2y-6=0$ denotes a straight line. The general equation of a straight line is $ax+by+c=0$ where a, b, c are some constant real numbers. We study here some special forms in which the equation of a straight line can be expressed.

1. Slope of a line.

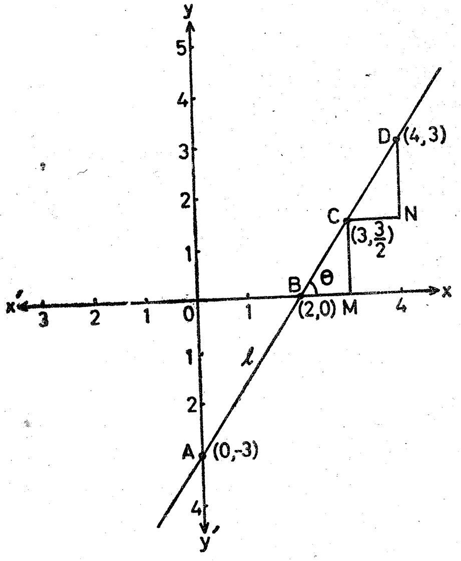

Fig. 7.1

Draw the graph of the equation $3x-2y-6=0$. It is a straight line l as shown in fig. 7.1. The points $A(0, -3)$, $B(2, 0)$, $C(3, \frac{3}{2})$, $D(4, 3)$ all lie on the line l. Now take any two of these points and draw lines parallel to the axes as shown in the figure.

The change in x from B to C is $3-2$ or 1 and

the „ „ y „ „ is $\frac{3}{2}-0$ or $\frac{3}{2}$

The change in y from one position of a point on the line to another is denoted by δy (read delta y) and the similar change in x by δx.

For any directed segment like BC, $\frac{\delta y}{\delta x}$ in the case of this line is $\frac{3}{2}$. This can also be seen from the fact that the $\triangle$s BCM, CDN are similar.

$\therefore \frac{CM}{BM}=\frac{DN}{CN}$. δy is the change in y and not $\delta \times y$. Similarly δx is the change in x and not $\delta \times x$.

Thus $\frac{\delta y}{\delta x}$ *or* $\frac{y_2-y_1}{x_2-x_1}$ *is constant for any line segment.*

If the line is parallel to the x-axis, y_2-y_1 is 0 in all cases and if the line is parallel to the y-axis, $x_2-x_1=0$. In the former case $\frac{\delta y}{\delta x}$ is 0 and in the latter case it is not defined.

If the line makes an angle θ with the x-axis, we see that $\frac{\delta y}{\delta x}$ is $\tan\theta$. $\frac{\delta y}{\delta x}$ is called the slope of the line and θ is the angle of inclination of the line with the x-axis. The slope may be thought of as a measure of the steepness of the line. The greater the angle, the steeper the line is. The slope of a line is also called its *gradient.*

Examples: The slope of the line segment joining $A(-5, -3)$ and $B(2, 1)$ is $\frac{y_2-y_1}{x_2-x_1}=\frac{1+3}{2+5}=\frac{4}{7}$.

The slope of the line segment joining $(-3, 2)$ and $(-5, 3)$ is $\frac{3-2}{-5+3}=\frac{1}{-2}=-\frac{1}{2}$. In the first case the angle of

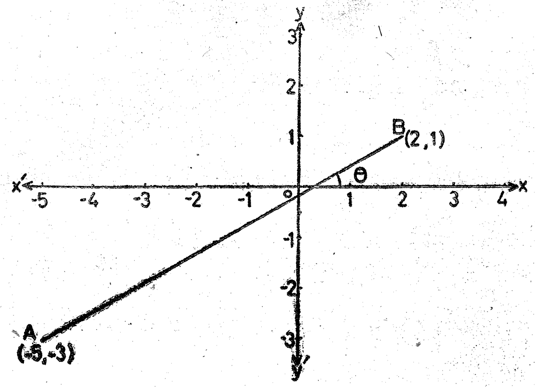

Fig. 7.2

inclination is $< \frac{\pi}{2}$ and in the second case it is $> \frac{\pi}{2}$ but $< \pi$.

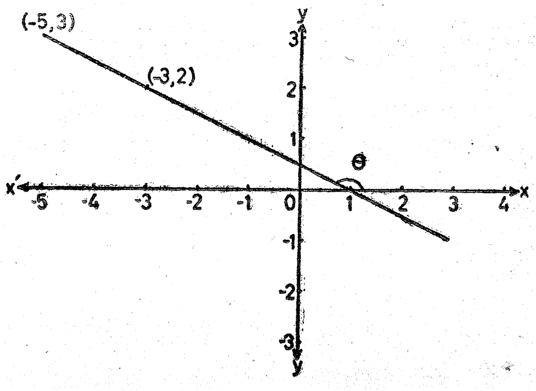

Fig. 7.3

2. The slope form of the equation of a line.

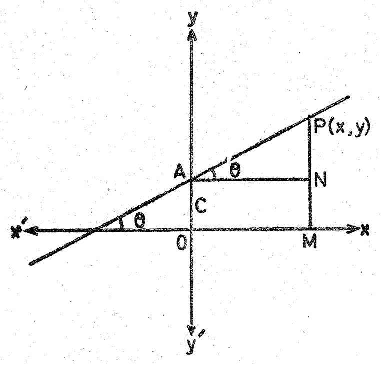

Fig. 7.4

In this figure (Fig. 7.4), *P* is any point on the line and its coordinates are x, y. c is the intercept the line makes on the y-axis. $PM=y$ and $PN=y-c$ and $AN=x$. Let m be the slope of the line.

$$\frac{PN}{AN}=\tan\theta=m=\frac{y-c}{x}$$

$\therefore y-c=mx$ or $y=mx+c$ is the equation of the line.

If the line passes through the origin, the equation becomes $y=mx$.

3. Line through a fixed point (x_1, y_1) with slope m.

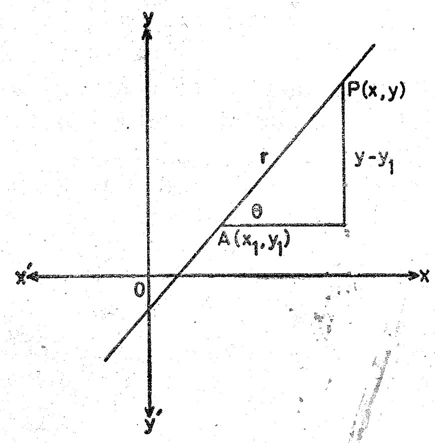

Fig. 7.5

Let $P(x,y)$ be any point on the line. Then its slope is $\frac{y-y_1}{x-x_1}=m$ or

$$y-y_1=m(x-x_1).$$

In this figure let the distance from the fixed point (x_1, y_1) to the variable point (x, y) be r.

We have $r\cos\theta=x-x_1$
and $r\sin\theta=y-y_1$

$$\therefore \frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r.$$

These are the parametric forms of the equation of a straight line.

4. Line through two given points (x_1, y_1) and (x_2, y_2).

The slope $=\dfrac{y_2-y_1}{x_2-x_1}=m$

Let (x, y) be any other point on the line.

Then $\dfrac{y-y_1}{x-x_1}=\dfrac{(y_2-y_1)}{(x_2-x_1)}=m$

This may also be written as

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}.$$

5. Intercept form of the equation of a line.

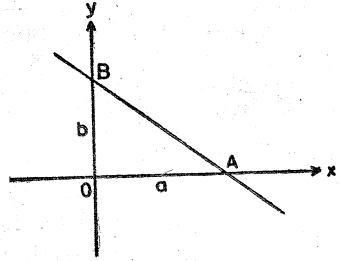

Fig. 7.6

If the line cuts the axes at A and B at distance of a and b from the origin, as shown in this figure, the coordinates of A and B are $(a, 0)$ and $(0, b)$ respectively.

The equation of the line is (see the form 4 above)

$$\frac{y-0}{b-0}=\frac{x-a}{0-a} \text{ or } \frac{y}{b}=\frac{x}{-a}+1 \text{ i.e. } -\frac{x}{a}+1$$

$$\therefore \frac{x}{a}+\frac{y}{b}=1 \text{ is the equation.}$$

6. Conditions for parallelism and perpendicularity.

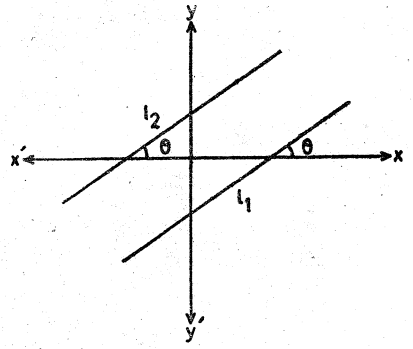

Fig. 7.7

(a) Parallelism.

If l_1 and l_2 are two parallel lines, they make the same angle with the x-axis.
$\therefore$ the slope is the same in both the cases.

If m_1 and m_2 are the slopes of two lines, then $m_1=m_2$. The two straight lines $ax+by=c$ and $a'x+b'y=c'$ are parallel if $\dfrac{a}{a'}=\dfrac{b}{b'}$ or $ab'=a'b$. For, the slopes are $-\dfrac{a}{b}$ and $-\dfrac{a'}{b'}$.

$$\therefore -\frac{a}{b}=-\frac{a'}{b'} \text{ or } ab'=a'b.$$

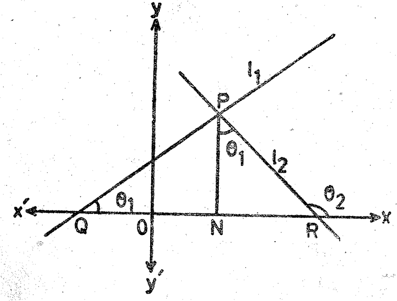

Fig. 7.8

(b) Perpendicularity.

Let l_1 and l_2 be the two lines intersecting at P. Draw $PN \perp x$-axis. If θ_1 and θ_2 are the angles made by the lines with the x-axis.

$\angle NPR$=compliment of $\angle QPN=\theta_1$

Now tan $\theta_1=\dfrac{NR}{PN}$ and tan $\theta_2=-\dfrac{PN}{NR}$, as $\angle PRN=\pi-\theta_2$

$$\therefore\ m_1m_2=-\frac{NR}{PN}\cdot\frac{PN}{NR}=-1$$

Thus the condition for two lines to be perpendicular is $m_1m_2=-1$.

The two straight lines $ax+by=c$ and $a'x+b'y=c'$ are perpendicular to each other if $aa'+bb'=0$; for the slope of $ax+by=c$ is $-\dfrac{a}{b}$ and the slope of $a'x+b'y=c'$ is $-\dfrac{a'}{b'}$.

$$m_1m_2=\left(-\frac{a'}{b'}\right)\cdot\left(-\frac{a}{b}\right)=-1$$

$$\therefore\ aa'=-bb' \text{ or } aa'+bb'=0.$$

Exercise 1

1-5. In the following graphs, find δx, δy and $\dfrac{\delta y}{\delta x}$:

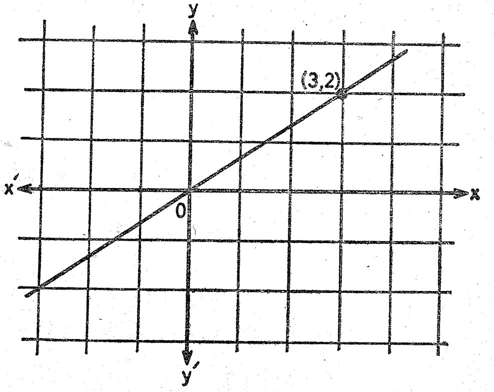

Fig. 7.9

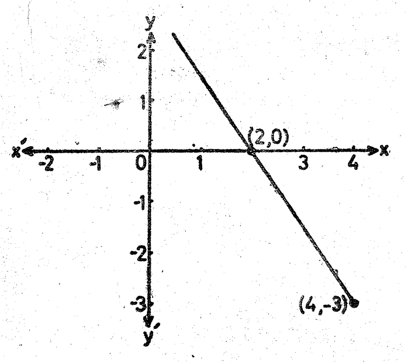

Fig. 7.10

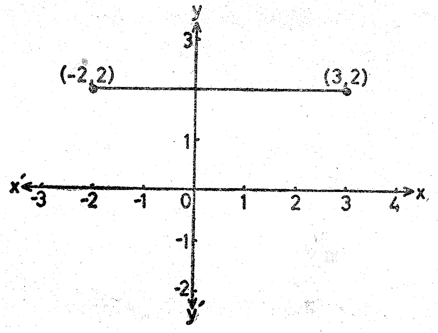

Fig. 7.11

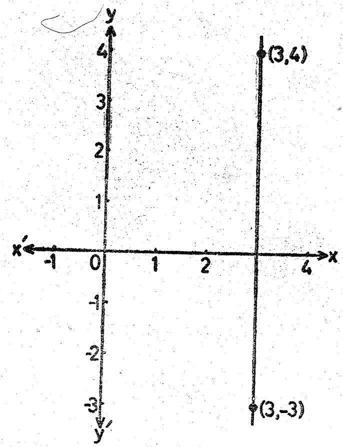

Fig. 7.12

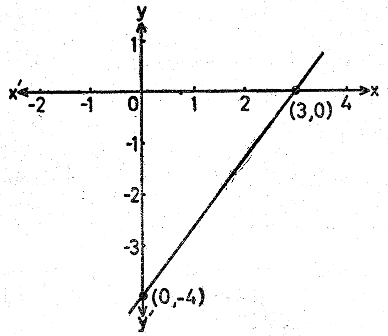

Fig. 7.13

Find the slopes of the line segments joining pairs of points with the coordinates as follows:

6. (5, 0), (—2, 0). y_2-y_1 **7.** (0, 5), (3, 5).

8. (3, —2), (3, 3). x_2-x_1 **9.** (—7, —3), (4, 8).

10. (9, —4), (13, —6). **11.** (8, 5), (8, 7).

12. Show that the three points $A(-4, 3)$, $B(2, 1)$ and $C(5, 0)$, lie on the same straight line. $x_1\ y_1$ $x_3\ y_3$

[*Hint:* Show that the slope of AB=slope of BC.]

13. Given the points $A(1, -1)$, $B(3, 3)$, $C(5, 7)$, find the slopes of AB, BC. Why are they collinear?

14. Given the points $A(6, 5)$, $B(1, 3)$, and $C(5, -7)$, find the slopes of AB and BC. Why is $AB \perp BC$?

15. Show that the points (2, 3), (4, 7), and (6, 1) are the vertices of a right-angled triangle.

16. Show that $AB \parallel CD$ in the following cases:

(i) $A(3, 1)$, $B(4, 3)$, and $C(1, 5)$, $D(-1, 1)$

(ii) $A(-1, 1)$, $B(2, -1)$ and $C(1, 1)$, $D(-2, 3)$.

17. Show that the four points $A(2, 6)$, $B(5, -1)$, $C(-4, 0)$ and $D(-7, 7)$, are the vertices of a parallelogram.

18. By transforming the equation $2x+3y-5=0$ to $y=mx+c$, find its slope.

19. What is the slope of the line $x+2y-5=0$? What is the slope of a line perpendicular to it?

20. Show that $2x-3y=5$ and $3x+2y=7$ are the equations of two straight lines perpendicular to each other.

21. Find the equation of a line through (2, 3) which makes an angle of 45° with the x-axis.

22. The slope of a line is $\frac{1}{3}$. Find its equation, if it passes through $(2, -2)$.

23. Find the equation of a line through (3,4) parallel to $y=\frac{1}{2}x+3$.

24. Find the equation of a line through (1, 5) perpendicular to the line $2x=3y$.

25. Find the equation of a line through $(2, -1)$ perpendicular to the line $3x+4y+7=0$.

26. Find the equation to the lines through

(i) (2, 3), (3, 2), (ii) (3, 5), (2, 2),

(iii) $(5, -1)$, $(-2, 1)$, (iv) $(-2, -3)$, $(3, 2)$,

(v) $(0, b)$, $(a, 0)$.

27. Find the equation to the lines through the origin parallel to the line joining $(1, -4)$ and $(3, 2)$.

28. Find the equations to the line through the origin perpendicular to the line joining (2, 1) and $(-2, -3)$.

29. Find the equation to the line through (1, 2) (i) parallel, (ii) perpendicular to the line $x=2y$.

30. Find the equations of the line which is the perpendicular bisector of the segment joining $A(1, 3)$, and $B(3, -5)$.

31. Find the equation of a line that intercept lengths of 2 and 3 units on the axes of x and y. What is the equation of a line through the origin perpendicular to the first one?

32. The intercepts of a straight line on the axes of x and y are -3 and 5. What is its equation? Show that the point (3, 10) lies on it.

33. Write down the equations of the lines through (1, 1) (i) parallel to $3x+5y=7$ and (ii) perpendicular to $y=4x$.

34. Write down the equation of a line through (2, 3) and perpendicular to $2x-5y=7$.

35. Find the equation whose x-intercept is $-\frac{3}{7}$, and which is perpendicular to the line $3x+4y=5$.

7. The equation of a line in the perpendicular form.

If the length of the perpendicular from the origin to the line and the angle this perpendicular makes with the x-axis are known, the equation of the line can be found. If p is the perpendicular distance and α is the angle OA makes with the x-axis, the coordinates of A are $(p \cos \alpha, p \sin \alpha)$. Also the slope of OA is $\tan \alpha$.

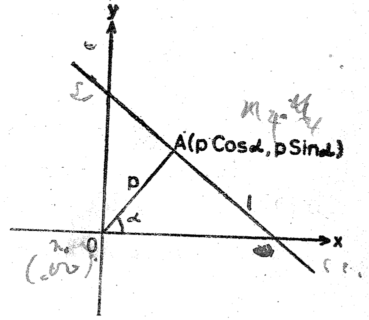

Fig. 7.14

$\therefore$ the slope of the line l is $-\cot \alpha$ as $mm^1=-1$.

$\therefore$ the equation of the line l is $y-p \sin \alpha$

$$=-\cot \alpha(x-p \cos \alpha)$$

$$\therefore y-p \sin \alpha+\frac{\cos \alpha}{\sin \alpha}(x-p \cos \alpha)=0$$

$$\therefore x \cos \alpha+y \sin \alpha-p \sin^2 \alpha-p \cos^2 \alpha=0$$

$\therefore x \cos \alpha+y \sin \alpha=p$ is the equation of the line l.

8. Perpendicular from the origin to a given line.

Any equation of a line $ax+by+c=0$ can be transformed into the form $x\cos\alpha+y\sin\alpha=p$. The latter is known as the *normal form.*

Suppose the equation of the given line is $3x-4y+5=0$. If $3x-4y+5=0$ and $x\cos\alpha+y\sin\alpha-p=0$ represent the same line, the coefficients of x, y and the constant must be proportional.

$\therefore \dfrac{\cos\alpha}{3}=\dfrac{\sin\alpha}{-4}=\dfrac{-p}{5}=\dfrac{\sqrt{\cos^2\alpha+\sin^2\alpha}}{\sqrt{3^2+4^2}}=\pm\frac{1}{5}$. Taking the negative sign, we have $\tan\alpha=-\frac{4}{3}$, $\sin\alpha=+\frac{4}{5}$ and $\cos\alpha=-\frac{3}{5}$.

$\therefore p=+1$. It is usual to make p positive.

If $ax+by+c=0$ is the equation of the given line,

$$\frac{\cos\alpha}{a}=\frac{\sin\alpha}{b}=\frac{-p}{c} \quad \therefore \tan\alpha=\frac{b}{a} \text{ and } \sin\alpha=\frac{b}{\sqrt{a^2+b^2}}$$

$$\therefore p=\frac{-c}{\sqrt{a^2+b^2}}$$

In order to keep the perpendicular from the origin to any line positive, we should see that c is positive. Then the origin is on the positive side of the line.

9. Perpendicular from (x_1, y_1) to $ax+by+c=0$.

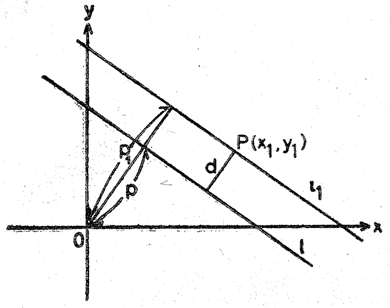

Fig. 7.15

Draw a line through (x_1, y_1) parallel to $ax+by+c=0$. Since the slope of the line is $-\dfrac{a}{b}$, the equation of the parallel line l_1 is $(y-y_1)=-\dfrac{a}{b}(x-x_1)$ or $ax+by-(ax_1+by_1)=0$.

From the result of the preceding article, we have p_1 the perpendicular distance from the origin to the line $l_1=-\dfrac{ax_1+by_1}{\sqrt{a^2+b^2}}$.

p the perpendicular to the given line $ax+by+c=0$ is $\dfrac{c}{\sqrt{a^2+b^2}}$. If d is the perpendicular distance from $P(x_1, y_1)$ to the line l i.e. $ax+by+c=0$

$$\begin{aligned} d=(p_1-p) &= \frac{-(c+ax_1+by_1)}{\sqrt{a^2+b^2}} \\ &= \frac{-(ax_1+by_1+c)}{\sqrt{a^2+b^2}} \end{aligned}$$

Here d is negative as p is positive. $P(x_1, y_1)$ and the origin are on the opposite sides of the given line.

The usual convention of sign is to make c positive and take positive roots. Then the origin is on the positive side of the line and any perpendicular from the side of the origin will be positive and from the opposite side will be negative.

Example 1: Show that the points (2, 3) and (5, −3) are on the opposite sides of the line $5x-12y=52$.

Let us take the origin on the positive side of the line.

Then the equation can be changed into $-5x+12y+52=0$.

The perpendicular from (2, 3) is $\dfrac{-10+36+52}{\sqrt{5^2+12^2}}=\dfrac{78}{13}=6$

$\therefore$ This point is on the positive side of the line. The perpendicular from (5, −3) to the line is

$$\frac{-5\cdot5+12\cdot(-3)+52}{\sqrt{5^2+12^2}}=\frac{-25-36+52}{13}=\frac{-11}{13}.$$

$\therefore$ the point (5, −3) is on the negative side.

Check up by drawing the line and plotting the two points.

Example 2: Find the equations of the bisectors of the angles formed by the line $x+3y+9=0$ and $13x-9y-27=0$.

A bisector of an angle is the locus of points or the set of points, at equal distances from the two rays forming the angle.

Write the second equation as $-13x+9y+27=0$ so that the origin may be on the positive side of the line.

Let l_1 and l_2 be the given lines and the side of each of these lines on which the origin lies is positive. Let $P(x, y)$ be any point of the locus. Then P in the figure is either on the negative side of both l_1 and l_2 or on the positive side of l_1 and l_2.

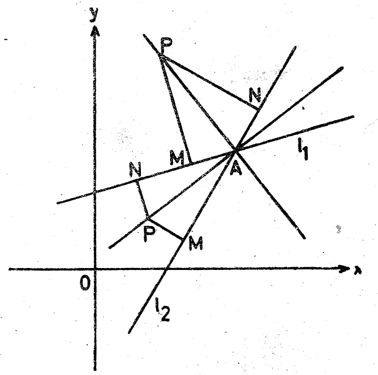

Fig. 7.16

$\therefore$ the equation of that bisector is

$$\frac{x+3y+9}{\sqrt{3^2+1^2}}=\frac{-13x+9y+27}{\sqrt{13^2+9^2}}.$$

$1^2+3^2=10$

$13^2+9^2=250=25\times 10$

$$\therefore\ \frac{x+3y+9}{\sqrt{10}}=\frac{-13x+9y+27}{5\sqrt{10}} \quad \text{or}$$

$5(x+3y+9)=-13x+9y+27$

$\therefore\ 18x+6y+18=0$ *i.e.* $3x+y+3=0$

If we take any point on the other bisector, the perpendiculars will be of opposite signs. In the figure the perpendicular on l_1 is positive and the perpendicular on l_2 is negative.

$\therefore$ the equation of the other bisector is

$$\frac{x+3y+9}{\sqrt{10}}=-\frac{-13x+9y+27}{5\sqrt{10}} \quad \text{or}$$

$5(x+3y+9)-13x+9y+27=0.$

$-8x+24y+72=0$ *i.e.* $x-3y-9=0.$

$\therefore\ x-3y-9=0$ and $3x+y+3=0$ are the equations of the two bisectors.

Exercise 2

1. Reduce the equations to the form $x\cos\alpha+y\sin\alpha=p$ in the following examples and find the distance of the line from the origin.

(i) $x-2y=0$

(ii) $3x+4=0$

(iii) $3x+4y=5$

(iv) $3x-4y=7$

(v) $2x-3y+5=0$

(vi) $x-\sqrt{3}\,y+6=0$

(vii) $x-y+5\sqrt{2}=0$

(viii) $5x+12y-13=0$

2. Find the distance between the parallel lines:

(i) $2x+3y-6=0;\ \ 2x+3y+5=0$

(ii) $3x-5y+2=0;\ \ 3x-5y-5=0$

(iii) $-5x+12y+13=0;\ \ 12y=5x+13$

3. Find the lengths of the following perpendiculars ignoring the signs:

(i) from $(1, 3)$ to $3x-4y+1=0$,

(ii) from $(1, 5)$ to $2x+3y-8=0$,

(iii) from $(-3, 2)$ to $y-2x+6=0$,

(iv) from $(3, -5)$ to $y=3x$,

(v) from $(6, 8)$ to $y=2$.

4. Find the perpendicular distance to the line $5x+12y-25=0$ from:

(i) $(3, 0)$, (ii) $(2, -5)$, (iii) $(7, 2)$, (iv) $(-5, 3)$, (v) $(4, 1)$. Which points lie on the same side of the line as the origin?

5. Find the distance from the point $(5, 7)$ to the line whose intercepts on the x and y axes are 2 and 1 respectively.

6. Find the perpendicular distance from the point $(2, -3)$ to the line of slope 1 through the point $(1, 1)$.

7. Find the distance from the point $(3, 5)$ to the line of slope $\frac{2}{5}$ which passes through the point $(2, -3)$.

8. Find the distance from the origin to the line joining the points $(3, 5)$ and $(-1, 2)$.

9. Find the distance from $(2, 3)$ to the line joining points $(1, 1)$ and $(5, -3)$.

10. Show that the points $A(2, -5)$ and $B(-1, 4)$ are equidistant from the line $3x+y+5=0$. Are A and B on the same or opposite side of the line? Prove that AB is parallel to the given line.

11. Show that the line $2x+3y-6=0$ is the perpendicular bisector of the line segment joining the origin and the point $(3, 2)$.
[*Hint.* Find the distances of the two points from the line and also show that the middle point of the segment joining the two given points lies on the given line.]

12. Find the distance between the two parallel lines $5x+12y-30=0$ and $5x+12y+22=0$.

13. Find the equation of the lines parallel to $x+3y=10$ to which the perpendicular distances from $(-1, 2)$ are $\pm\sqrt{10}$ units.

Find the equations of the two bisectors of the angles made by the lines.

14. $x-2y+3=0$ and $x-3y-5=0$.

15. $2x-y+5=0$ and $3x+y+7=0$.

16. $x-3y+5=0$ and $3x-y-9=0$.

17. $x-y+20=0$ and $17x-7y+20=0$.

18. $5x+12y-52=0$ and $4x-3y+34=0$.

19. Find the lengths of the altitude of the triangle whose vertices are $(2, 1)$, $(4, -3)$ and $(-4, 0)$.

20. Find the equations of the internal bisectors of the angles of the triangles whose sides are

$$3x-4y=2,\ 3x+4y=10 \text{ and } y=0.$$

10. Lines through the intersection of two lines.

Let us take any two intersecting lines l_1 and l_2 whose equations are $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$.

Consider now the equation

$$a_1x+b_1y+c_1+k(a_2x+b_2y+c_2)=0 \text{ where } k \text{ is any constant.} \quad (1)$$

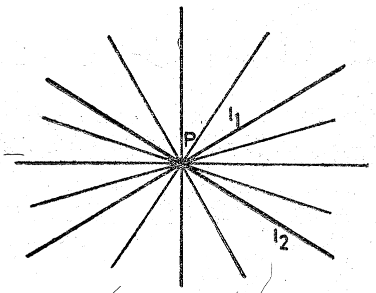

Fig. 7.17

Since this is an equation of the first degree, it represents a straight line. The point of intersection P of l_1 and l_2 lies on this also, as its coordinates satisfy the equations $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$. We can assign any value to k and still the equation (1) is satisfied by the coordinates of the point of intersection. Geometrically, the equation (1) represents different straight lines through P, for different values of k. Suppose one of these lines passes also through another given point. By substituting the coordinates of this point we find the numerical value of k and then the line is fixed. This enables us to write down the equation of a line through the point of intersection, without actually finding the coordinates of P.

Example 1*:* Find the equation of the line through the intersection of the lines $2x+3y-5=0$ and $3x-5y+7=0$ and through the point $(3, 2)$.

Let $2x+3y-5+k(3x-5y+7)=0$ be the required line.

Since the coordinates $x=3$, $y=2$ must satisfy this equation, we have

$$6+6-5+k(9-10+7)=0 \text{ or } 7+6k=0.$$

$\therefore k=-\frac{7}{6}$

$\therefore 6(2x+3y-5)-7(3x-5y+7)=0$

$\therefore 12x+18y-30-21x+35y-49=0$

$\therefore -9x+53y-79=0$

$9x-53y+79=0$ is the equation.

The other point should not be on any of the given lines; for instance, if the point is (1, 1) we have $0+5k=0 \quad \therefore k=0$. This means the required line is l_1 itself.

Example 2: Find the equation of the line passing through the intersection of the lines $4x-5y-5=0$ and $13x-15y=1$ and parallel to the line $2x-5y=7$.

Let the line be $4x-5y-5+k(13x-15y-1)=0$

i.e., $x(4+13k)-y(5+15k)-5-k=0$. The slope of this line must be the same as that of $2x-5y-7=0$.

$$\therefore \text{slope}=\frac{4+13k}{5+15k}=\frac{2}{5} \quad \therefore 5(4+13k)=10(1+3k)$$

$$\therefore 4+13k=2+6k \quad \therefore 7k=2-4=-2$$

$$\therefore k=\frac{-2}{7}.$$

The equation of the line required is

$4x-5y-5-\frac{2}{7}(13x-15y-1)=0$

i.e. $7(4x-5y-5)=2(13x-15y-1)$

i.e. $2x-5y=33$.

Exercise 3

1. Find the equation of the line through the intersection of the lines $2x-5y+7=0$ and $3x+2y+5=0$, having the slope of $-\frac{1}{2}$.

2. Find the equation of the line through the intersection of the lines $x\ +3y+5$ and $5x+y+3=0$, having an intercept 2 on the y-axis.

x k
(2 0)

3. Find the equation of the line through the intersection of lines $8x+y+12=0$ and $7x-3y-2=0$, having an intercept -3 on the x-axis.

4. Find the equation of the line through the intersection of the lines $3x-8y+12=0$ and $2x+3y-7=0$ and through the point $(-3, 1)$.

5. Find the equation of the lines through the intersection of the lines $x+3y+9=0$ and $3x-9y-27=0$ and parallel to the line $2x+3y+5=0$.

6. Find the equation of the line through the intersection of the lines $2x-7y+10=0$ and $3x+5y-5=0$ and through the point (2, 1).

7. Find the equation of the line through the intersection of the lines $2x+3y-7=0$ and $3x-4y=0$ and having a slope $-\frac{2}{7}$.

8. Find the equation of the line through the intersection of the line $x-8y+11=0$ and $4x-7y+3=0$ and perpendicular to $3x+2y+5=0$.

9. Find the equation of the line through the intersection of the lines $2x-3y-5=0$ and $6x+9y+11=0$ and perpendicular to the first of the given lines.

10. Find the equations of the line through the intersection of the lines $2x+3y+9=0$, $5x-2y-3=0$ and parallel to the x-axis.

11. Find the equation of the line through the intersection of the lines $3x+5y+7=0$, $2x+3y+4=0$ and parallel to the y-axis.

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12. Find the equation of the line through the intersection of the lines $3x-2y-3=0$, $2x-y-1=0$ and bisecting the segment joining the points (1, 5) and (−3, 3).

11. The equation of a locus.

A locus of points is the same thing as a set of points. Since every set of points is called a figure, it is a curve satisfying given conditions in two dimensional geometry and a surface or a curve satisfying given conditions in three-dimensional geometry. For our purposes here, a locus of points is a curve satisfying given mathematical conditions. We often refer to a particular point on the locus, which means a member of the set of points. The locus can often be expressed by an algebraic equation. A point lies on a curve if and only if its coordinates satisfy the equation of the curve.

Example 1: Find the equation of the locus of points which are equidistant from two given points $A(1, 2)$ and $B(-2, 5)$.

Let $P(x, y)$ be any point on the locus.

Its distance from A is $\sqrt{(x-1)^2+(y-2)^2}$ and that from B is $\sqrt{(x+2)^2+(y-5)^2}$.

Since these distances are equal from the given condition,

$$\sqrt{(x-1)^2+(y-2)^2}=\sqrt{(x+2)^2+(y-5)^2}$$

Squaring both sides and collecting like terms, we have

$$(x-1)^2+(y-2)^2=(x+2)^2+(y-5)^2$$

$$x^2+y^2-2x-4y+5=x^2+y^2+4x-10y+29$$

$$\therefore\ -6x+6y=24 \quad \textit{i.e.,}\ -x+y=4.$$

The equation $y-x-4=0$ states the geometrical condition which is satisfied by any of its points and further expresses the same condition analytically in terms of the coordinates x, y.

very difficult
but important

Exercise 4

1. In the example illustrated above, draw the graph of the equation of the locus. Join the two given points. What is the relation of the two points to the locus? Is the point (3, 7) on the locus? What is the distance of this point from the two given points?

2. Find the equations of the locus of points equidistant from
   (i) (−3, 7) and (2, −3)
   (ii) (5, 0) and (−3, −3)
   (iii) (5, 2) and (−7, 2)
   (iv) (0, 0) and (3, −4).

3. Find the equation of the locus of points equidistant from the x-axis and the y-axis.

4. Find the equation of the locus of points which are at a distance of 5 cm. from the origin.

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5. Find the equation of the locus of points at a constant distance of 2 cm from the point (1, 2). Do the points (3, 2), (2, 1), (1, 0), (4, 0) lie on the curve?

6. Find the equation of the locus of points equidistant from the x-axis and the point (1, 2). Does the point (3, 2) lie on the locus?

Find the equation of the locus of a point $P(x, y)$ satisfying the following conditions:

7. The point is twice as far from the origin as far from the point (1, 1).

8. The point is half as far from (5, 0) as from the y-axis.

9. The point is three times as far from (2, 0) as from (−2, 0).

10. The distance of the point from (3, 4) is equal to twice its distance from the origin.

11. The sum of the squares of the distances from the point to the points $(0, a)$ and $(0, -a)$ is $4a^2$.

12. Given the points $A(0, 2)$ and $B(0, -2)$. Find the equation of the locus of $P(x, y)$ if $PA+PB=6$.

[Solution. $PA^2=x^2+(y-2)^2$ and $PB^2=x^2+(y+2)^2$

$$\therefore\ \sqrt{x^2+(y-2)^2}+\sqrt{x^2+(y+2)^2}=6.$$

Squaring both sides, we have

$$x^2+(y-2)^2+x^2+(y+2)^2=2\sqrt{x^2+y^2+4-4y}\,\sqrt{x^2+y^2+4+4y}=36.$$

$$\therefore\ 2(x^2+y^2)+8-36=2\sqrt{(x^2+y^2+4)^2-16y^2}$$

Dividing by 2 and squaring again

$$(x^2+y^2-14)^2 = (x^2+y^2+4)^2-16y^2$$

$$(x^2+y^2)^2-28(x^2+y^2)+196 = (x^2+y^2)^2+8(x^2+y^2)+16-16y^2$$

$$\therefore\ 36(x^2+y^2)-16y^2=180$$

$\therefore\ 9x^2+5y^2=45$ is the equation of the locus.]

13. The sum of the distances from (0, −1) and (0, 1) to a point on the locus is $=\sqrt{2}$.

14. Find the locus of the vertex of an isosceles triangle if the end points of the base are (3, 0) and (−3, 0).

15. The end points of one of the equal sides of an isosceles triangle are (1, −3) and (2, −1). Find the locus of the third vertex.

16. Given the points $A(2, 0)$ and $B(-2, 0)$. Find the equation of the locus of a point P such that $PA \perp PB$. Does the locus pass through A and B?

[*Hint.* Assume x_1, y_1 as a point on the locus. Write down the condition for PA to be perpendicular to PB. Then to make the condition applicable for all points of the locus, write x and y in place of x_1, y_1.]

17. The end points of the base of a triangle ABC are $A(1, 3)$ and $B(3, 5)$. Find the locus of the vertex C, if the altitude is 2.

18. The base of a triangle is the segment joining (2, 5) (−2, 3); the altitude is 1 unit. Find the locus of the vertex.

12. Area of a triangle.

Consider a triangle with vertices $A(x_1, y_1)$ $B(x_2, y_2)$ and $C(x_3, y_3)$.

Draw AL, BM, $CN \perp$ the x-axis as shown in the figure.

$AL = y_1$; $BM = y_2$; and $CN = y_3$.

$LM = x_2 - x_1$; $NM = x_2 - x_3$ and $LN = x_3 - x_1$.

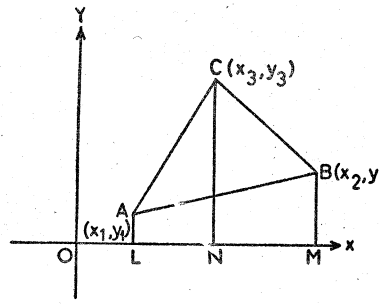

Fig. 7.18

In this figure, the area of $\triangle ABC$ = area of trapezium $ALNC$ + area of trapezium $CNMB$ − area of a trap. $ALMB$.

$$= \tfrac{1}{2}(x_3 - x_1)(y_1 + y_3) + \tfrac{1}{2}(x_2 - x_3)(y_2 + y_3) - \tfrac{1}{2}(x_2 - x_1)(y_2 + y_1)$$

$$= \tfrac{1}{2}\{(x_2 - x_3)(y_2 + y_3) + (x_3 - x_1)(y_3 + y_1) - (x_2 - x_1)(y_1 + y_2)\}$$

$$= \tfrac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Here we have assumed the result in Geometry that the area of a trapezium is equal to $\frac{1}{2}$ the altitude × sum of the lengths of the parallel sides. For different positions of A, B, C, the same result is obtained.

The area will be positive or negative according as the direction we take along the perimeter of the triangle is counter-clockwise or clockwise. In either case, the numerical value of the area is the same.

Another form which is convenient to use for the area of a triangle is the form of an array.

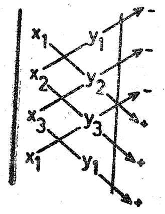

Fig. 7.19

Note that the first row of the array is repeated.

To find the area of a polygon, divide it into triangles and find the sum of the areas of the triangles. If the area of a triangle is zero, the three points are collinear. If (x_1, y_1), (x_2, y_2) and (x_3, y_3) are three collinear points, then

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Example: Find the area of the triangle with vertices $(-1, -4)$, $(6, -3)$ and $(-2, -1)$.

$$\frac{1}{2}\begin{vmatrix} -1 & -4 & 1 \\ 6 & -3 & 1 \\ -2 & -1 & 1 \end{vmatrix} = \tfrac{1}{2}\{-1(-3+1)+4(6+2)+(-6-6)\}$$
$$=\tfrac{1}{2}\times 22 = 11$$

Using the array,

$$\frac{1}{2}\begin{vmatrix} -1 & -4 \\ 6 & -3 \\ -2 & -1 \\ -1 & -4 \end{vmatrix} = \tfrac{1}{2}\{3-6+8+24-6-1\}$$
$$=\tfrac{1}{2}\times 22 = 11.$$

Exercise 5

Find the area of the triangle or quadrilateral with the vertices as given.

1. The origin, $(3, 0)$ and $(2, 5)$.

2. $(1, 2)$, $(2, 3)$, $(3, 4)$.

3. $(2, -1)$, $(-3, 2)$, $(3, 1)$.

4. $(3, 2)$, $(1, -4)$, $(5, 3)$.

5. $(2, 3)$, $(-2, 5)$, $(-4, -2)$.

6. $(1, -3)$, $(3, 4)$, $(5, 2)$.

7. (a, b), (b, c), (c, a).

8. $(0, 0)$, $(0, 1)$, $(2, 3)$, $(1, 2)$.

9. $(-1, 2)$, $(-3, 2)$, $(3, 4)$, $(2, 5)$.

10. $(2, -3)$, $(1, -2)$, $(-2, 3)$, $(1, 1)$.

11. Find the altitude of the triangle $A(3, 1)$, $B(0, 5)$, $C(-4, 3)$ drawn from B. Calculate the area and the length AC.

12. If the area of a triangle with vertices $(0, y)$, $(-2, 1)$ and $(2, 1)$ is 4 square units, find the value of y.

13. By using the area formula show that the points $(1, 4)$, $(4, -2)$, $(-5, 16)$ are collinear.

14. Find the equation of the line passing through $(3, 2)$ and $(2, 3)$ by equating the area of the triangle $(3, 2)$, $(2, 3)$ and (x, y) to zero.

15. Two vertices of a triangle are $(4, 3)$, $(5, 6)$ and the area of the triangle is 3 units. Find the locus of the third vertex.

13. Application to Pure Geometry.

Some theorems and riders of Elementary Geometry can be proved analytically by the results of Analytical Geometry.

1. Prove that the medians of a triangle meet at a point G which divides each median in the ratio $2:1$. Let the coordinates of A, B, C be (x_1, y_1), (x_2, y_2) and (x_3, y_3). In the figure it is required to prove that the medians AD, BE, CF meet at G such that $AG:GD=BG:GE=CG:GF=2:1$.

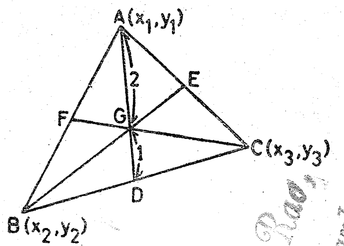

Fig. 7.20

In the median AD, the coordinates of D are

$$\frac{x_2+x_3}{2},\ \frac{y_2+y_3}{2}$$

If we divide AD at G in the ratio $2:1$, the coordinates of G are

$$x = \frac{\dfrac{2(x_2+x_3)}{2}+x_1}{2+1} = \frac{x_1+x_2+x_3}{3}$$

$$y = \frac{\dfrac{2(y_2+y_3)}{2}+y_1}{2+1} = \frac{y_1+y_2+y_3}{3}.$$

From this one result, the theorem can be deduced. For, if we divide the segments BE and CF internally in the ratio $2:1$, the points of division will have the same coordinates, as that of AD. This follows from the symmetrical result of the coordinates of G. You can also verify it by actually finding the coordinates of the other points of division. Since the coordinates of the points of division of the three segments are the same, the three points are one and the same. Hence the theorem is proved.

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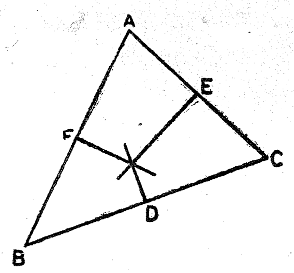


Fig. 7.21

2. Prove that the perpendicular bisectors of the three sides of a triangle meet at a point.

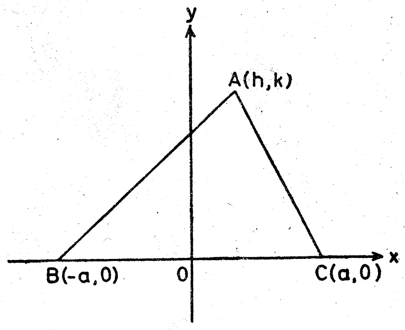


Fig. 7.22

In such cases we can place the axes in a convenient form without loss of generality. Here we can take the perpendicular bisector of BC as the y-axis. Take $BC=2a$. Then the coordinates of B, C will be $(-a, 0)$ and $(a, 0)$. Let the coordinates of A be (h, k). The equation of the perpendicular bisector of BC is $x=0$. Now to find the equation of the perpendicular bisector of AC we note that the slope of AC is $\frac{k}{h-a}$ so that the slope of a line $\perp$ AC is $\frac{-(h-a)}{k}$. Also the coordinates of the midpoint of AC are $\frac{h+a}{2}, \frac{k}{2}$.

$\therefore$ the equation of the perpendicular bisector of AC is

$$\left(y-\frac{k}{2}\right)+\frac{(h-a)}{2}\left(x-\frac{h+a}{2}\right)=0$$

i.e., $k(2y-k)+2x(h-a)-(h^2-a^2)=0\ldots.(1)$

The equation of the perpendicular bisector of AB is obtained by putting $-a$ in place of a in (1), as there is no other change in the coordinates.

$$\therefore\ k(2y-k)+2x(h+a)-(h^2-a^2)=0\ldots.(2)$$

$$\text{and } x=0\ldots.(3)$$

(1), (2), (3), are the equations of the three perpendicular bisectors.

By substituting $x=0$ in (2), we find that the perpendicular bisectors of AB and BC meet in a point

$$\left(0, \frac{k^2+h^2-a^2}{2k}\right)$$

Similarly by substituting $x=0$ in (1), we find that the perpendicular bisectors of AC and BC meet in a point

$$\left(0, \frac{k^2+h^2-a^2}{2k}\right)$$

Since both these points of intersection have the same coordinates, they are one and the same point.

$\therefore$ the perpendicular bisectors are concurrent.

Note. Here had we taken the coordinates of A, B, C as (x_1, y_1), (x_2, y_2) and (x_3, y_3), it would have been more difficult to work. If we take one side as the x-axis, it is simple to work and there is no loss of generality, as we are free to take any two perpendicular lines as the coordinate axes.

Exercise 6

1. Show that the line joining the midpoints of any two sides of a triangle is parallel to the third side.

2. Prove that the lines joining the midpoint of the opposite sides of a quadrilateral bisect each other. [Let (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4) be the coordinates of the four vertices of the quadrilateral.]

3. The midpoint of the hypotenuse of a right-angled triangle is equidistant from the three vertices of the triangle. [Let the rays forming the right angle be x and y axes.]

4. In an isosceles triangle ABC, $AB=AC$. If D and F are the midpoints of AC and AB, show that $BD=CE$.

5. Prove that a quadrilateral whose diagonals bisect each other is a parallelogram.

[Take one side of a quadrilateral as the x-axis.]

6. A rectangle whose diagonals are equal is a square.

7. The diagonals of a rhombus are perpendicular.

[Take the vertices as $(0, 0)$, (x_1, y_1), (x_1+a, y_1), $(a, 0)$ where $x_1^2+y_1^2=a^2$]

8. Prove that a quadrilateral whose diagonals bisect each other at right angles is a rhombus.

[Take the diagonals as axes.]

9. The line segments joining the midpoints of adjacent sides of a quadrilateral form a parallelogram.

10. In any triangle, the sum of the squares of any two sides is equal to twice the square of the median to the third side, plus half the square of the third side. [Take the base as the x-axis with its midpoint as the origin.]

11. The distance between the midpoints of the non-parallel sides of a trapezium is half the sum of the parallel sides.

CHAPTER 8

The Circle

1. A circle is the locus of points in a plane such that the line segments joining them with a fixed point O in the plane are equal to a given segment. The fixed point O is the centre of the circle. The length of the given line segment is denoted by r units. Let (a, b) be the centre and r the radius of the circle. Then the point (x, y) shall be at a fixed distance r from (a, b) if

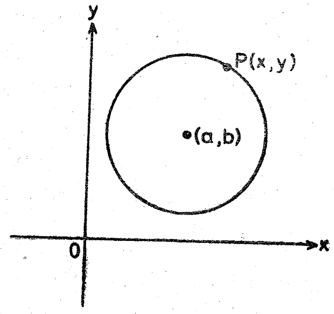


Fig. 8.1

$$\sqrt{(x-a)^2+(y-b)^2}=r$$

i.e., $(x-a)^2+(y-b)^2=r^2$. (1)

This is the equation of a circle. If the centre is the origin itself, the equation will be reduced to the form $x^2+y^2=r^2$.

The equation of any circle $(x-a)^2+(y-b)^2=r^2$, becomes $x^2+y^2-2ax-2by+a^2+b^2-r^2=0$. There are two characteristics of the general equation of the circle viz. (1) the coefficients of x^2 and y^2 are the same and (2) there is no term in xy. Hence the general equation of a circle takes the form

$$x^2+y^2+2gx+2fy+c=0. \quad (2)$$

If the equation is, for instance, $2x^2+2y^2+3x+5y+3=0$, we can rewrite it by dividing throughout by 2, as $x^2+y^2+\frac{3}{2}x+\frac{5}{2}y+\frac{3}{2}=0$. Here $f=\frac{5}{4}$, $g=\frac{3}{4}$ and $c=\frac{3}{2}$. To find the centre and the radius of the circle (2), we may write it as

$(x+g)^2+(y+f)^2=g^2+f^2-c$. Comparing this equation with (1), we see that the centre is $(-g, -f)$ and the radius is $\sqrt{g^2+f^2-c}$.

Since there are three arbitrary constants involved in the general equation of a circle, we need three equations connecting these constants. An equation arises when the circle is subjected to some geometrical condition. For instance, if the circle passes through a given point, it gives rise to one equation. Hence if a circle passes through three given points, the circle is uniquely determined.

If $g=0$, the centre is on the y-axis; if $f=0$, the centre is on the x-axis, and if $c=0$, the circle passes through the origin.

Example 1: Find the equation of the circle passing through (1, 1), (2, 3) and (2, −1). Find also the coordinates of the centre and the radius.

The general equation of a circle is $x^2+y^2+2gx+2fy+c=0$. The coordinates (1, 1), (2, 3) and (2, −1) satisfy this equation as the points lie on the circle. The three simultaneous equations in f, g, c arising out of these conditions are

$$2g+2f+c+2=0 \qquad (1)$$
$$4g+6f+c+13=0 \qquad (2)$$
$$4g-2f+c+5=0 \qquad (3)$$

Adding (1) and (3), we have $6g+2c+7=0$. (4)

Multiply (3) by 3 and add (2) to it. Then we have $16g+4c+28=0$.

$\therefore$ $4g+c+7=0$. (5). Solving the two equations (4) and (5), we have $g=-3\frac{1}{2}$ and $c=7$. Substituting these values in (1) we have $f=-1$. $\therefore$ the equation of the circle is $x^2+y^2-7x-2y+7=0$ *i.e.*, $(x-\frac{7}{2})^2+(y+1)^2=\frac{25}{4}$.

$\therefore$ $(3\frac{1}{2}, -1)$ is the centre and $2\frac{1}{2}$ is the radius.

Example 2: Show that the perpendicular bisector of a chord of a circle passes through the centre.

Let $(2x_1, 2y_1)$ and $(2x_2, 2y_2)$ be the two points on the circle $x^2+y^2=4a^2$. The centre is the origin and the radius is $2a$.

Since points $(2x_1, 2y_1)$ and $(2x_2, 2y_2)$ lie on the circle, $4x_1^2+4y_1^2=4a^2$ and $4x_2^2+4y_2^2=4a^2$.

$\therefore$ $x_1^2+y_1^2=a^2$ and $x_2^2+y_2^2=a^2$. The midpoint of the chord is (x_1+x_2, y_1+y_2). The slope of the chord is $\frac{y_2-y_1}{x_2-x_1}$ and the slope of a perpendicular to the chord is $\frac{-(x_2-x_1)}{y_2-y_1}$.

$\therefore$ the equation of the perpendicular bisector of the chord is $y-(y_1+y_2)=\frac{-(x_2-x_1)}{y_2-y_1}(x-x_1-x_2)$

This passes through the origin, if $x=0$, $y=0$ satisfies it, that is, if

$$-(y_2-y_1)(y_1+y_2)=(x_1+x_2)(x_2-x_1)$$

or $-y_2^2+y_1^2=x_2^2-x_1^2$ which is true

as $x_1^2+y_1^2=x_2^2+y_2^2=a^2$

Hence the result follows.

Example 3: Find the equation of the circle on the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ as diameter. Let $P(x, y)$ be any point on the circle. Since the angle in a semicircle is a right angle, $AP \perp BP$. The slopes of these lines are

$$\frac{y-y_1}{x-x_1} \text{ and } \frac{y-y_2}{x-x_2}$$

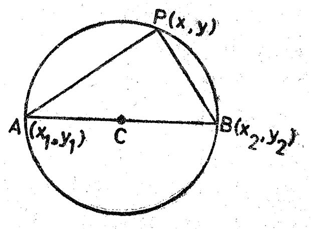


Fig. 8.2

Since $mm'=-1$, $\dfrac{y-y_1}{(x-x_1)}\times\dfrac{y-y_2}{x-x_2}=-1$

$\therefore$ $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$ *is the required equation of the circle on the line segment joining* (x_1, y_1) *and* (x_2, y_2) *as diameter.*

If the equation of a circle reduces to the form $(x-a)^2+(y-b)^2=0$, the circle becomes a point-circle, as its radius is zero.

Note that when the equation of a circle reduces to the form $(x-a)^2+(y-b)^2=-c$, where c is positive, the circle is not real.

Exercise 1

Find the equation of the circle in each of the following cases:

1. Centre is (1, 1) and radius is 3.
2. Centre is (2, 3) and radius is 1.
3. Centre is (2, -2) and radius is 3.
4. Centre is (1, -3) and radius is 2.
5. Centre is (-2, -3) and radius is 5.
6. Centre is (4, -1) and passes through the origin.
7. Centre is (2, -5) and passes through the origin.
8. The origin and the point (3, 1) are the ends of a diameter.
9. The points (2, 5) and (-4, 1) are the ends of a diameter.
10. The points (8, -2) and (-2, 6) are the ends of a diameter.

Find the centre and radius of each of the following circles and draw it.

11. $x^2+y^2-4x=0$.

12. $x^2+y^2-8y=0$.

13. $x^2+y^2-4x+2y=0$.

14. $x^2+y^2+4x+4y=0$.

15. $x^2+y^2+4x+3=0$.

16. $x^2+y^2+6y-7=0$.

17. $x^2+y^2+2x+4y-4=0$.

18. $x^2+y^2+6x-4y-12=0$.

19. $x^2+y^2-12x+4y+15=0$.

20. $x^2+y^2+3x+5y-\frac{1}{2}=0$.

21. $4x^2+4y^2+4x+8y+1=0$.

22. $3x^2+3y^2+6x-2y+4=0$.

23. Find the equation of the diameter of the circle $x^2+y^2+6x-2y-15=0$, which, when produced, passes through the point (-8, 2).

24. Find the equation of the circle passing through (-1, -1), (0, 1) and (2, 0).

25. Find the equation of the circle passing through (2, 3), (-2, 1) and (0, 5).

26. Find the equation of the circle passing through (1, 1), (2, 3) and (-3, 1).

27. Find the equation of the circle passing through $A(3, 1)$, $B(1, -1)$ and $C(-1, 1)$ by using the fact that the perpendicular bisector of a chord passes through the centre of the circle.

28. Find the equation of the circle circumscribing the triangle formed by the coordinate axes and the line $2x+3y=6$.

29. Find the equation of the circle circumscribing the triangle formed by the lines $x-y+2=0$, $x+y-8=0$ and $x-2y+4=0$.

30. Show that the equation $(x-5)^2+(y-12)^2=169$ represents a circle which passes through the origin. Find the coordinates of the other end of the diameter through the origin. Find also the coordinates of the points P, Q at which the circle cuts again the x and y axes. Show that PQ is a diameter.

31. Find the equation of a circle, if (7, 8) and (1, 0) are the ends of one of its diameters. If the circle cuts the x-axis at A and B and the y-axis at C and D show that $AB=CD$.

2. The tangent to a circle at a given point.

Let the equation of the circle be $x^2+y^2=r^2$ and $P(x_1, y_1)$ be any point on it. If R is another point near P on the circle, the limiting position of the line PR as R approaches or tends to P is called the tangent at P to the circle.

It is proved in Pure Geometry that the tangent at any point P on the circle is perpendicular to OP, where O is the centre.

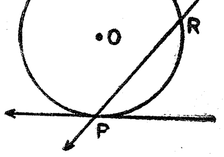


Fig. 8.3

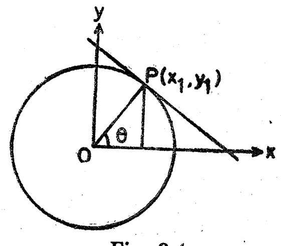


Fig. 8.4

The slope of $OP = \tan\theta = \frac{y_1}{x_1}$.

The slope of the tangent at P

$$= \frac{-x_1}{y_1}, \text{ as } mm' = -1.$$

$\therefore$ the equation of the tangent to the circle $x^2+y^2=r^2$ at $P(x_1, y_1)$ is

$$y - y_1 = \frac{-x_1}{y_1}(x - x_1)$$

i.e., $y_1(y-y_1)+x_1(x-x_1)=0$

$\therefore yy_1+xx_1-x_1{}^2-y_1{}^2$ or $xx_1+yy_1-r^2=0$

is the equation of the tangent, since $x_1{}^2+y_1{}^2=r^2$.

Any point P on the circle has its coordinates $x_1=r\cos\theta$, $y_1=r\sin\theta$ as seen in the figure, where θ is the angle of inclination of OP and r is the radius of the circle. $r\cos\theta$, $r\sin\theta$ are referred to as the parametric coordinates of a point on the circle $x^2+y^2=r^2$.

Consequently the tangent at the point 'θ' is

$$xr\cos\theta + yr\sin\theta = r^2$$

or

$$x\cos\theta + y\sin\theta = r.$$

Hence the equation of the tangent at any point (x_1, y_1) or θ is

$$xx_1+yy_1=r^2$$

or $\quad x\cos\theta+y\sin\theta=r.$

If $ax+by+c=0$ is a tangent to the circle $x^2+y^2=r^2$, the perpendicular distance from the centre to the given line must be equal to the radius numerically.

$$\therefore \frac{-c}{\sqrt{a^2+b^2}}=r \text{ or } r^2(a^2+b^2)=c^2.$$

In a similar manner the student may show that the equation of the tangent to the circle $x^2+y^2+2gx+2fy+c=0$ at the point $P(x_1, y_1)$ is $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$.

Example 1: Find the equation of the tangent to the circle

(i) $x^2+y^2=25$ at $P(3, 4)$ and

(ii) $x^2+y^2-4x+6y-3=0$ at the point $(2, 1)$.

[(i) $xx_1+yy_1=25$. Here it is $3x+4y=25$.

(ii) $(2, 1)$ lies on the given circle, for, $4+1-8+6-3=0$

The equation of the tangent is $xx_1+yy_1-2(x+x_1)$
$+3(y+y_1)-3=0.$

If $x_1=2, y_1=1$, the equation of the tangent is
$2x+y-2(x+2)+3(y+1)-3=0)$

$\therefore 4y-4=0$ or $y-1=0$.

Geometrically we infer from this equation that the tangent is parallel to the x-axis as the slope is zero. The line joining the centre and the given point must be parallel to the y-axis.]

Example 2: Find the equations of the tangents to the circle $x^2+y^2=13$, where the circle cuts the line $x-5y+13=0$.

[The points of intersection of the given line and the given circle are found by substituting $x=5y-13$ in the equation $x^2+y^2=13$ and solving the quadratic equation in y.

$(5y-13)^2+y^2=13$ or $26y^2-130y+156=0$

i.e., $y^2-5y+6=0$ or $(y-3)(y-2)=0$

When $y=3$, $x=5y-13=2$ and when $y=2$, $x=-3$.

$\therefore$ the points of intersection are $(2, 3)$, $(-3, 2)$

$\therefore$ equations of tangents at these points are

$2x+3y=13$ and $-3x+2y=13$.]

3. Length of a tangent from a point to a circle.

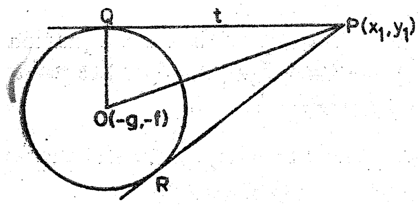


Fig. 8.5

The general equation of a circle is $x^2+y^2+2gx+2fy+c=0$ (1)
If O is the centre of this circle its coordinates are $(-g, -f)$.

From Pure Geometry, $PQ \perp OQ$

$$\therefore PQ^2=PR^2=OP^2-r^2$$
$$=(x_1+g)^2+(y_1+f)^2-r^2$$
$$=x_1^2+2gx_1+g^2+y_1^2+2fy_1+f^2-r^2$$
$$=x_1^2+y_1^2+2gx_1+2fy_1+c \text{ as } r^2=g^2+f^2-c$$

We note that the expression for t^2 is the left member of equation (1) above with x and y replaced by x_1 and y_1 respectively.

The length of a tangent from (x_1+y_1) to the circle $x^2+y^2=r^2$ is $\sqrt{x_1^2+y_1^2-r^2}$. If t^2 is positive, the point (x_1, y_1) lies outside the circle. If it is zero, the point is on the circle. If it is negative, the point is within the circle.

Example 3: Find the length of the tangent from the point (3, 5) to the circle.

(i) $x^2+y^2-1=0$ and

(ii) $x^2+y^2+4x+8y-5=0$

(i) $t^2=9+25-1=33 \quad \therefore t=\sqrt{33}$

(ii) $t^2=9+25+12+40-5=81 \quad \therefore t=9$.

Example 4: Show that $y=mx\pm r\sqrt{1+m^2}$ is a tangent to the circle $x^2+y^2=r^2$ for each finite value of m.

If we draw the perpendicular to the lines from the origin, *i.e.*, the centre of the circle, its length

$$=\frac{\pm r\sqrt{1+m^2}}{\sqrt{1+m^2}}=\pm r$$

Since the length of the perpendicular from the centre is equal to the radius, the given line touches the circle. Hence the parallel lines $y=mx\pm r\sqrt{1+m^2}$ are tangents to the circle. Show that the points of contact are

$$\left(\frac{mr}{\sqrt{1+m^2}}, \frac{-r}{\sqrt{1+m^2}}\right) \text{ and } \left(\frac{-mr}{\sqrt{1+m^2}}, \frac{r}{\sqrt{1+m^2}}\right)$$

Show also that they are the ends of a diameter of the circle.

This form of the equation of tangents is useful in finding the equation of the tangents to a circle from an external point.

Example 5: Find the equations of the tangents from the point (8, 2) to the circle $x^2+y^2=34$.

The equation of any straight line through (8, 2) is of the form $(y-2)=m(x-8)$ or $y=mx+2-8m$, where m is the slope.

Comparing this with $y=mx\pm r\sqrt{1+m^2}$ or in this case with $y=mx\pm\sqrt{34(1+m^2)}$ which are equations of the tangents with slope m to the given circle, we have $2-8m=\pm\sqrt{34(1+m^2)}$. Squaring both the sides and rearranging the terms, we have $34(1+m^2)=(2-8m)^2$ or $30m^2-32m-30=0$. $\therefore 15m^2-16m-15=0$
i.e., $(5m+3)(3m-5)=0$. $\therefore m=-\frac{3}{5}$ or $\frac{5}{3}$. Hence there are two values of m.

The equations of the two tangents are $y=-\frac{3}{5}x+2+\frac{24}{5}$ and $y=\frac{5}{3}x+2-\frac{40}{3}$, substituting these values of m in the equation $y=mx+2-8m$.

Simplifying these equations, the two required equations are $5y+3x-34=0$ and $3y-5x+34=0$.

4. Pairs of orthogonal circles.

When any two circles intersect at a point P, the tangents to the two circles at P make an angle. This angle is said to be the angle at which the two circles intersect. Two circles are said to be *orthogonal*, when each cuts the other at right angles. In this case, the tangent to each becomes a radius of the other, as the angle between the two tangents is a right angle.

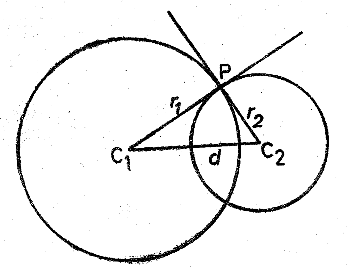


Fig. 8.6

Here C_1P is a tangent to the circle with centre C_2 and radius r_2 and C_2P is a tangent to the circle with centre C_1 and radius r_1. C_1PC_2 is a right-angled triangle, with the right angle at P.

Let the circle with C_1 as centre and r_1 as radius be

$$x^2+y^2+2g_1x+2f_1y+c_1=0 \quad (1)$$

and the circle with C_2 as centre and radius r_2 be

$$x^2+y^2+2g_2x+2f_2y+c_2=0 \quad (2)$$

d, the distance between the two centres is given by $d^2=r_1^2+r_2^2$, by the theorem of Pythagoras.

Now $r_1^2=g_1^2+f_1^2-c_1$, $r_2^2=g_2^2+f_2^2-c_2$ and

$d^2=(g_1-g_2)^2+(f_1-f_2)^2$, since the centres are $(-g_1, -f_1)$ and $(-g_2, -f_2)$

$$\therefore\ d^2=r_1^2+r_2^2=g_1^2+f_1^2-c_1+g_2^2+f_2^2-c_2$$
$$=(g_1-g_2)^2+(f_1-f_2)^2$$
$$=g_1^2+f_1^2+g_2^2+f_2^2-2g_1g_2-2f_1f_2$$
$$\therefore\ -(c_1+c_2)=-2(g_1g_2+f_1f_2)$$

$\therefore\ 2g_1g_2+2f_1f_2=c_1+c_2$ is the condition that the two circles are *orthogonal i.e.*, cut each other at right angles.

Example 6: Show that the two circles $x^2+y^2-2x-2y=7$ and $3x^2+3y^2-8x+29y=0$ are orthogonal.

The equations of the two circles may be written as

$x^2+y^2-2x-2y-7=0.$ Here $g_1=-1, f_1=-1, c_1=-7.$

$x^2+y^2-\frac{8}{3}x+\frac{29}{3}y+0=0.$ $g_2=-\frac{4}{3}, f_2=\frac{29}{6}, c_2=0.$

$\therefore\ 2(g_1g_2+f_1f_2)=\frac{8}{3}-\frac{29}{3}=-\frac{21}{3}=-7=c_1+c_2$

$\therefore$ the condition is satisfied.

5. Radical axis of two circles.

The radical axis of two circles is defined as the locus of points such that the tangents from each of them to the two circles are equal. There are three cases.

(a) The two given circles may intersect.

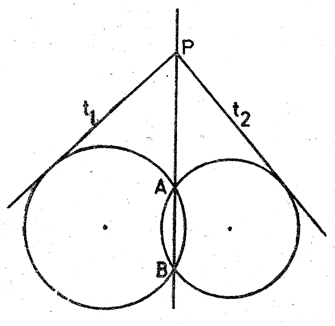
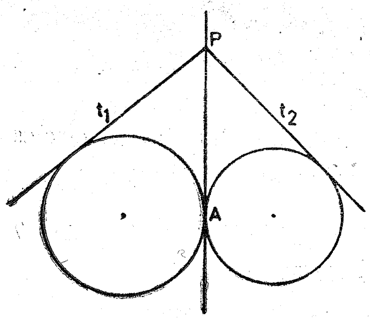


Fig. (8.7a)

The radical axis here is the common chord. $t_1=t_2$.

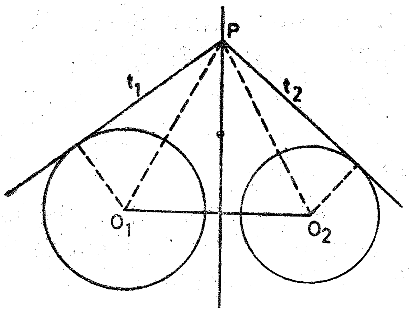
(b) The two circles may touch.



Here the radical axis is the common tangent at A.

Fig. 8.7 (b)

(c)



Here the radical axis is a line perpendicular to the line joining the two centres.

Fig. 8.7 (c)

Let the circles be $x^2+y^2+2g_1x+2f_1y+c_1=0$ and $x^2+y^2+2g_2x+2f_2y+c_2=0$. Let x_1, y_1 be any point on the locus. $t_1^2=t_2^2$.

$$\therefore x_1^2+y_1^2+2g_1x+2f_1y+c_1=x_1^2+y_1^2+2g_2x_1+2f_2y_1+c_2$$

$$\therefore 2g_1x_1+2f_1y_1+c_1=2g_2x_1+2f_2y_1+c_2$$

Since (x_1, y_1) is any point on the locus, the coordinates may be replaced by (x, y) in the condition.

$$\therefore 2(g_1-g_2)x+2(f_1-f_2)y+c_1-c_2=0$$

is the equation of the radical axis.

Notice that

$x^2+y^2+2g_1x+2f_1y+c_1+k(x^2+y^2+2g_2x+2f_2y+c_2)=0$ is a circle passing through the points of intersection of the given circles. For different values of k, there will be different circles passing through the two points of intersection. The set of circles for different values of k is referred to as the system of *coaxal circles*. Every pair of these circles has the same radical axis.

Example 7: Find the equation of a circle which passes through the point $(-2, 3)$ and passes through the intersection of the circle $x^2+y^2-5=0$ and $x^2+y^2+2x-4y-2=0$.

The set of circles passing through the intersection of the given circles is $x^2+y^2-5+k(x^2+y^2+2x-4y-2)=0$.

We can find the circle of this set, which passes through the point $(-2, 3)$ by substituting these coordinates in the equation and finding the value of k.

$\therefore 4+9-5+k(4+9-4-12-2)=0$ or $8-5k=0$.

$\therefore k=\frac{8}{5}$

$\therefore$ the equation of this circle is $5(x^2+y^2-5)$
$+8(x^2+y^2+2x-4y-2)=0$.

i.e. $13x^2+13y^2+16x-32y-41=0$.

Example 8: Find the equation of the radical axis of the circles $2x^2+2y^2+3x-4y-1=0$ and $3x^2+3y^2-5x-2y=0$.

$2x^2+2y^2+3x-4y-1+k(3x^2+3y^2-5x-2y)=0$ is the system. For the radical axis, we must so choose k that the coefficients of x^2 and y^2 vanish. $\therefore k=-\frac{2}{3}$.

$$3x+\tfrac{10}{3}x-4y+\frac{4y}{3}-1=0$$

$$\therefore 19x-8y-3=0.$$

Exercise 2

Find the equation of the tangent to each of the following circles at the given point:

1. $x^2+y^2=25$ at $(-3, 4)$.
2. $x^2+y^2=3$ at $(\sqrt{3}, 1)$.
3. $x^2+y^2=5$ at $(2, 1)$.
4. $x^2+y^2=9$ at $(2, \sqrt{5})$.
5. $x^2+y^2=11$ at $(2\sqrt{2}, \sqrt{3})$.
6. $x^2+y^2=17$ at $(1, -4)$.
7. $25x^2+25y^2=16$ at $\left(\frac{2\sqrt{3}}{5}, -\frac{2}{5}\right)$
8. $x^2+y^2-4x-6y+9=0$ at $(2, 5)$.
9. $x^2+y^2-2x-8y+15=0$ at $(2, 3)$.
10. $x^2+y^2+6x+4y-12=0$ at the points where the circle cuts the y-axis.
11. $x^2+y^2=20$ at the points where the circle cuts the line $x=4$.
12. $x^2+y^2=18$ at the points where the line $y=x$ cuts the circle.
13. Find the length of the tangent from $(3, -2)$ to the circle $x^2+y^2=8$.
14. Find the length of the tangent from $(5, 2)$ to the circle $x^2+y^2-4x+6y=0$. Draw the figure.
15. If t is the length of the tangent from $P(x_1, y_1)$ to the circle $x^2+y^2=9$, what is the value of t^2? If $t^2=16$, what is the locus of P? Draw the figures of the circle and the locus.
16. Find the locus of points from which tangents to the circle $x^2+y^2+2x-4y=0$ are equal to 2 units in length.
17. Find the locus of points from which equal tangents of length $\sqrt{6}$ units can be drawn to the circle $2x^2+2y^2+3x-2y-5=0$. Show that $(1, -2)$ is a point on the locus.
18. Find out whether the following points lie on, without or within the circle $x^2+y^2+3x+5y-10=0$:
    $(2, -2)$, $(1, 1)$ and $(3, 2)$.

19. Show that the line $x=6$ is a tangent to the circle $x^2+y^2-2x+4y-20=0$. Find the equation of the diameter through the point of contact.
20. Show that the line $2x+3y+26=0$ is a tangent to the circle $x^2+y^2-4x-6y-104=0$. 3513
21. Find the equation of the tangent at the point $(-3, 4)$ to the circle $x^2+y^2=25$. Find also the equations of the two tangents parallel to the axis of y. If the first tangent intersects the other two at points A and B, show that AB subtends a right angle at the origin.
22. Find the equation of the circle which has its centre at the point $(4, -2)$ and touches the line $y=3x$.
23. Find the equations of the circle which has its centre at $(3, 5)$ and touches the line $3x+4y=19$.
24. A circle passes through the point $A(2, 5)$ and touches the y-axis at $B(0, 1)$. Show that the equation of the circle is $x^2+y^2-10x-2y+1=0$. [*Hint:* Find the equation of the perpendicular bisector of AB and that of the equation perpendicular to the y-axis at B. The intersection of the two lines gives the centre of the circle.]
25. Find the equation of the circle which passes through the point $(2, -1)$ and touches the line $3x-2y=6$ at the point $(4, 3.)$
26. Find the radical axis of the two circles
    (a) $x^2+y^2+6x+8y+5=0$ and $x^2+y^2=16$
    (b) $x^2+y^2-3x-2y-7=0$ and $x^2+y^2-7x-5y+2=0$.
27. Show that the radical axes of the three given circles given below taken in pairs are concurrent and so find the point through which they pass.
    (i) $x^2+y^2+3x-2y-4=0$, (ii) $x^2+y^2-2x-y-6=0$,
    (iii) $x^2+y^2-1=0$.
    The point is known as the *Radical Centre.*
28. Find the equations of the three radical axes of the three given circles taken in pairs and find the radical centre.
    (i) $x^2+y^2+3x+2y+1=0$, (ii) $x^2+y^2-x+6y+5=0$,
    (iii) $x^2+y^2+5x-8y+15=0$.

29. Prove that the circles $x^2+y^2-2x=0$ and $x^2+y^2+4y=0$ cut orthogonally.

30. Prove that the following pairs of circles cut orthogonally:

(i) $x^2+y^2+4x+6y+5=0$ and $x^2+y^2+6x-4y-5=0$

(ii) $x^2+y^2+8x+12y+40=0$ and $x^2+y^2+4x+10y+36=0$

(iii) $x^2+y^2+2gx+2fy+c=0$ and $x^2+y^2+2fx-2gy-c=0$.

END

PART III

TRIGONOMETRY

CHAPTER 9

The Trigonometric Functions

1. You are already familiar with the trigonometrical ratios. For any given angle θ (say, in radians) $\sin\theta$ and $\cos\theta$ are fixed and unique. $\tan\theta$, $\cot\theta$, $\operatorname{cosec}\theta$ and $\sec\theta$ are also fixed and unique except for some values of θ. Thus $\tan\theta$ and $\sec\theta$ are not defined for values $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ etc. and $\cot\theta$ and $\operatorname{cosec}\theta$ are not defined for $0, \pi, 2\pi, 3\pi$ etc., of θ.

2. Functions of complementary angles.

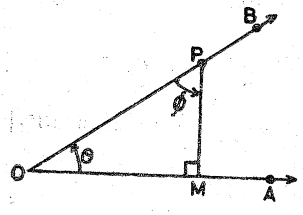


Fig. 9.1

Draw an angle θ at O with the rays OA and OB and $PM \perp OA$ from any point P on OB.

Now $\sin\theta = \dfrac{PM}{OP}$ and $\cos\theta = \dfrac{OM}{OP}$.

$\angle OPM = \frac{\pi}{2} - \theta$ in radians, as OPM is a right-angled triangle. If this angle is denoted by ϕ, we see that $\sin\phi = \dfrac{OM}{OP} = \cos\theta$ and $\cos\phi = \dfrac{PM}{OP} = \sin\theta$. But θ and ϕ are complementary angles; that is, $\theta+\phi=\frac{\pi}{2}$, a right angle.

$\therefore$ $\sin\theta=\cos\phi$ and $\cos\theta=\sin\phi$ where θ and ϕ are complementary angles. These relationships hold good for all angles θ and ϕ either positive or negative. This will be proved later.

3. Functions of supplementary angles.

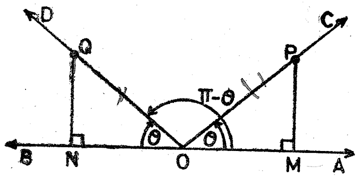


Fig. 9.2

On a straight line AB take a point O and make equal angles AOC and BOD as shown in the figure. Let each of these angles be θ in radians. Since $A\hat{O}B$ is a straight angle, $A\hat{O}Q=\pi-\theta$. Take P on OC and Q on OD such that $OP=OQ$. Draw $PM \perp OA$ and $QN \perp OB$. Now it's easily seen that Δ^s POM and QON are congruent so that $OM=ON$ and $PM=QN$ numerically.

$$\sin A\hat{O}Q=\sin(\pi-\theta)=\frac{QN}{OQ}=\frac{PM}{OP}=\sin\theta$$

$$\cos A\hat{O}Q=\cos(\pi-\theta)=\frac{ON}{OQ}=-\frac{OM}{OP}=-\cos\theta.$$

[∵ OM and ON are equal but are in opposite directions.]

$$\tan(\pi-\theta)=\frac{\sin(\pi-\theta)}{\cos(\pi-\theta)}=-\frac{\sin\theta}{\cos\theta}=-\tan\theta$$

Similarly cosec $(\pi-\theta)=$cosec θ, sec $(\pi-\theta)=-$sec θ and cot $(\pi-\theta)=-$cot θ.

This again will be found to be true for any value of θ.

4. Functions of $\pi+\theta$.

In this figure, the ray OC makes an angle of θ with OA and COD is a straight angle.

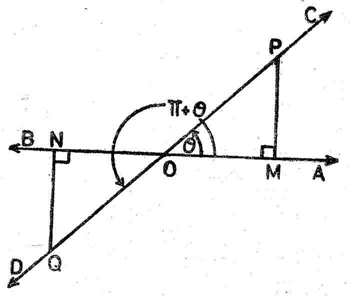


Fig. 9.3

∴ $A\hat{O}D=\pi+\theta$ in radians.

Let $OP=OQ$ and $PM \perp AB$ and $QN \perp AB$

The Δ^s POM and QON are congruent so that $ON=OM$, $OQ=OP$.

$QN=PM$. Taking the directions into consideration, $ON=-OM$ and $PM=-QN$. The rotating ray OC or OD has no direction.

$$\therefore \sin(\pi+\theta)=\frac{QN}{OQ}=-\frac{PM}{OP}=-\sin\theta$$

$$\cos(\pi+\theta)=\frac{ON}{OQ}=-\frac{OM}{OP}=-\cos\theta$$

$$\tan(\pi+\theta)=\frac{QN}{ON}=\frac{-PM}{-OM}=\frac{PM}{OM}=\tan\theta$$

Similarly cosec $(\pi+\theta)=-$cosec θ, sec $(\pi+\theta)=-$sec θ cot $(\pi+\theta)=$cot θ.

5. Functions of $\frac{\pi}{2}+\theta$

In this figure, $A\hat{O}C=\theta$ and $A\hat{O}D=\frac{\pi}{2}+\theta$ so that $OC \perp OD$. If P and Q are such that $OP=OQ$ and PM and $QN \perp AB$, we have

$$\angle NOQ=\pi-\left(\frac{\pi}{2}+\theta\right)=\frac{\pi}{2}-\theta,$$

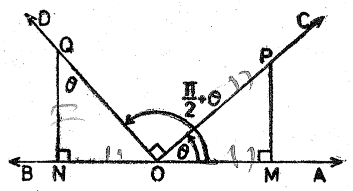


Fig. 9.4

so that $N\hat{Q}O=\theta$. ∴ Δ^s OPM and OQN are congruent.

∴ $ON=PM$ numerically and $QN=OM$.

$$\sin\left(\frac{\pi}{2}+\theta\right)=\frac{QN}{OQ}=\frac{OM}{OP}=\cos\theta$$

$$\cos\left(\frac{\pi}{2}+\theta\right)=\frac{ON}{OQ}=-\frac{PM}{OQ}\text{ as } ON \text{ is in the negative direction,}$$

$$=-\sin\theta$$

$$\tan\left(\frac{\pi}{2}+\theta\right)=\frac{\sin\left(\frac{\pi}{2}+\theta\right)}{\cos\left(\frac{\pi}{2}+\theta\right)}=-\cot\theta$$

Similarly $\operatorname{cosec}\left(\frac{\pi}{2}+\theta\right)=\sec\theta$

$$\sec\left(\frac{\pi}{2}+\theta\right)=-\operatorname{cosec}\theta$$

$$\cot\left(\frac{\pi}{2}+\theta\right)=-\tan\theta.$$

6. Functions of $2\pi\pm\theta$.

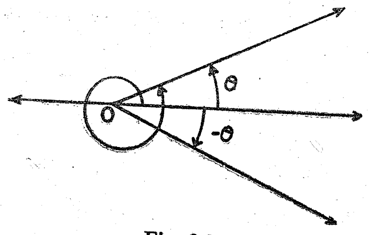


Fig. 9.5

From this figure, it is seen that $2\pi+\theta$ is the same angle as θ geometrically. $\therefore$ $\sin(2\pi+\theta)=\sin\theta$ and $\cos(2\pi+\theta)=\cos\theta$.

From this it is clear that all trigonometrical functions are *periodic* with the period 2π i.e. all trigonometric functions have the same values, when the independent variable θ is increased by 2π. It is also seen that $(2\pi-\theta)$ is the same angle in the figure as $-\theta$. Now we have seen in chapter 4, $\sin(-\theta)=-\sin\theta$ and $\cos(-\theta)=\cos\theta$.

$\therefore$ $\sin(2\pi-\theta)=-\sin\theta$, $\cos(2\pi-\theta)=\cos\theta$ and $\tan(2\pi-\theta)=-\tan\theta$ etc.

Example 1: Find $\cos\left(\frac{\pi}{2}-\theta\right)$, $\cos\left(\frac{\pi}{2}+\theta\right)$, $\cos(\pi+\theta)$, $\cos(\pi-\theta)$, and $\sin\left(\frac{3\pi}{2}+\theta\right)$ when $\sin\theta=\frac{3}{5}$.

Taking θ to be an acute angle, $\cos\theta=\frac{4}{5}$.

[$\because$ $\cos^2\theta+\sin^2\theta=1$, $\cos\theta=\pm\sqrt{1-\frac{9}{25}}=\pm\frac{4}{5}$, since θ is acute, $\cos\theta=\frac{4}{5}$.]

(i) $\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta=\frac{3}{5}$;

(ii) $\cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta=-\frac{3}{5}$

(iii) $\cos(\pi+\theta)=-\cos\theta=-\frac{4}{5}$

(iv) $\cos(\pi-\theta)=-\cos\theta=-\frac{4}{5}$

(v) $\sin\left(\frac{3\pi}{2}+\theta\right)=\sin\left(\pi+\frac{\pi}{2}+\theta\right)$

$$=-\sin\left(\frac{\pi}{2}+\theta\right)$$
$$=-\cos\theta$$
$$=-\frac{4}{5}.$$

Exercise 1

Express as a function of θ:

1. $\tan\left(\frac{\pi}{2}+\theta\right)$, $\tan\left(\frac{\pi}{2}-\theta\right)$, $\cos(3\pi+\theta)$, $\cos\left(\frac{3\pi}{2}+\theta\right)$.
2. $\tan(\pi-\theta)$, $\cot(\pi-\theta)$, $\sin(\pi+\theta)$ $\sec(\pi+\theta)$.
3. $\sin(2\pi-\theta)$, $\cos(2\pi-\theta)$, $\tan(2\pi-\theta)$, $\cot(2\pi-\theta)$.
4. $\sin(\pi+\theta)$, $\cos(\pi+\theta)$, $\tan(\pi+\theta)$, $\cot(\pi+\theta)$.
5. $\sin\left(\frac{3\pi}{2}-\theta\right)$, $\cos\left(\frac{3\pi}{2}-\theta\right)$, $\tan\left(\frac{3\pi}{2}-\theta\right)$, $\cot\left(\frac{3\pi}{2}-\theta\right)$

Find the values of

6. sin 120°, cos 120°, cos 150°, cos 210°, cos 240°.
7. sin 135°, cos 135°, sin 225°, cos 225°.
8. $\sin 3\pi$, $\cos\frac{3\pi}{4}$, $\cos\frac{11\pi}{4}$, $\sin\frac{7\pi}{4}$.
9. $\sin\frac{2\pi}{3}$, $\cos\frac{2\pi}{3}$, $\sin\frac{4\pi}{3}$, $\cos\frac{4\pi}{3}$.

10. $\cos \pi$, $\cos 2\pi$, $\cos 3\pi$, $\cos 4\pi$.

11. If $\tan 75° = 3.73$, find $\cot 15°$, $\tan 165°$ and $\tan 255°$.

12. Find the value of: $\dfrac{\cot \frac{\pi}{6} - \cot \frac{\pi}{4}}{\cot \frac{\pi}{6} + \cot \frac{5\pi}{4}}$

13. Prove that $\sin 670° = -\sin 50°$.

14. Prove that $\tan 325° = -\cot 55°$.

15. Prove that $\cos(-680°) = \cos 40°$.

Imp.

7. Simple identities.

We have already seen in chapter 4, that $\sin^2\theta + \cos^2\theta = 1$. This is true for any value of θ. Just as $(a+b)^2 = a^2 + 2ab + b^2$ is an algebraic identity, true for any numerical values of a and b, $\sin^2\theta + \cos^2\theta = 1$ is a trigonometric identity true for any numerical value of θ.

Similarly $\tan\theta \cot\theta = 1$ and $\sec\theta = \dfrac{1}{\cos\theta}$ are also identities except for the quadrantal values of θ.

Example 2: Prove that $\dfrac{\sin\theta}{\operatorname{cosec}\theta} + \dfrac{\cos\theta}{\sec\theta} = 1$

$$\frac{\sin\theta}{\operatorname{cosec}\theta} + \frac{\cos\theta}{\sec\theta} = \frac{\sin\theta}{\frac{1}{\sin\theta}} + \frac{\cos\theta}{\frac{1}{\cos\theta}}$$
$$= \sin^2\theta + \cos^2\theta$$
$$= 1.$$

Example 3: Prove that $\dfrac{\sin^3\phi + \cos^3\phi}{\sin\phi + \cos\phi} = 1 - \sin\phi\cos\phi$.

$$\sin^3\phi + \cos^3\phi = (\sin\phi + \cos\phi)(\sin^2\phi - \sin\phi\cos\phi + \cos^2\phi)$$
$$= (\sin\phi + \cos\phi)(1 - \sin\phi\cos\phi)$$ in accordance with the algebraic identity
$$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$

$$\therefore \frac{\sin^3\phi + \cos^3\phi}{\sin\phi + \cos\phi} = 1 - \sin\phi\cos\phi$$

Example 4: Prove that $\dfrac{\sec\alpha - \operatorname{cosec}\alpha}{\sec\alpha + \operatorname{cosec}\alpha} = \dfrac{\tan\alpha - 1}{\tan\alpha + 1}$

$$\frac{\sec\alpha - \operatorname{cosec}\alpha}{\sec\alpha + \operatorname{cosec}\alpha} = \frac{\frac{1}{\cos\alpha} - \frac{1}{\sin\alpha}}{\frac{1}{\cos\alpha} + \frac{1}{\sin\alpha}} = \frac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha} \quad \ldots\ldots (1)$$

(by multiplying the numerator and denominator by $\sin\alpha\cos\alpha$)

Now $$\frac{\tan\alpha - 1}{\tan\alpha + 1} = \frac{\frac{\sin\alpha}{\cos\alpha} - 1}{\frac{\sin\alpha}{\cos\alpha} + 1} = \frac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha} \quad \ldots\ldots (2)$$

Since (1) and (2) are the same, the identity holds good.

In solving problems on identities,

(1) reduce the right side to the left side or vice versa.

(2) or reduce each side to a third result as done in ex. 4 above.

It is understood that the values of the variable are such that they will not make any denominator zero.

Exercise 2

Prove each of the following identities:

1. $\sin\theta \sec\theta \cot\theta = 1$.

2. $\cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$.

3. $\dfrac{\cos^2\theta}{1 - \sin\theta} = 1 + \sin\theta$.

4. $\dfrac{1 - \cos\theta}{1 + \cos\theta} = (\operatorname{cosec}\theta - \cot\theta)^2$.

5. $\tan\theta \sin\theta = \sec\theta - \cos\theta$.

6. $\sec^2\theta - 1 = \sin^2\theta \sec\theta$.

7. $1 + \tan^2\theta = \dfrac{1}{1 - \sin^2\theta}$.

8. $\sin^2\theta(1 + \cot^2\theta) = 1$.

9. $\tan^2 \alpha\,(\operatorname{cosec}^2 \alpha - 1) = 1.$

10. $(1+\tan^2 \alpha)\,(1-\sin^2 \alpha) = 1.$

11. $\dfrac{\sec \alpha + \operatorname{cosec} \alpha}{1+\tan \alpha} = \operatorname{cosec} \alpha.$

12. $\dfrac{\sin \alpha}{\sec \alpha} = \dfrac{1}{\tan \alpha + \cot \alpha}.$

13. $\dfrac{1}{\sec^2 \alpha} + \dfrac{1}{\operatorname{cosec}^2 \alpha} = 1.$

14. $\tan \alpha + \cot \alpha = \sec \alpha \operatorname{cosec} \alpha.$

15. $\dfrac{1+\cos\theta}{1-\cos\theta} - \dfrac{1-\cos\theta}{1+\cos\theta} = 4 \cot\theta \operatorname{cosec}\theta.$

16. $(\tan\theta + \cot\theta)^2 = \sec^2\theta + \operatorname{cosec}^2\theta.$

17. $(1+\sec\theta)(1-\cos\theta) = \sec\theta \sin^2\theta.$

18. $\dfrac{\tan x}{\sec x + 1} + \dfrac{\tan x}{\sec x - 1} = 2 \operatorname{cosec} x.$

19. $\dfrac{1+\cot x}{\operatorname{cosec} x} = \dfrac{1+\tan x}{\sec x}.$

20. $\sin x \cos x\,(\tan x + \cot x) = 1.$

21. $\cos^4 \alpha - \sin^4 \alpha = 2\cos^2 \alpha - 1.$

22. $\tan^2\phi - \sin^2\phi = \sin^2\phi \tan^2\phi.$

23. $\sec^4\phi - \sec^2\phi = \tan^4\phi + \tan^2\phi.$

24. $\dfrac{1-2\cos^2\beta}{\sin\beta\cos\beta} = \tan\beta - \cot\beta.$

25. $\dfrac{2+\tan^2\beta}{\sec^2\beta} = 1+\cos^2\beta.$

26. $\dfrac{1+\tan^2\theta}{1+\cot^2\theta} = \dfrac{1-\cos^2\theta}{\cos^2\theta}.$

CHAPTER 10

Solution of Triangles

1. One of the practical uses to which trigonometry can be put is the solution of triangles; that is, we can determine the measurements of certain parts of a triangle, when the measurements of the other parts are given.

2. Right-angled triangle

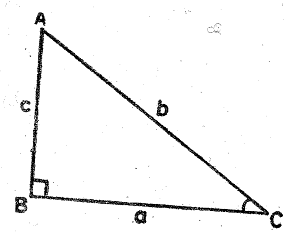


Fig. 10.1

Here if the angle C and the side a are given, the other parts are easily found or calculated. $A = 90° - C$. So all the three angles are known. We can find the sides AB and AC by using the relation $\tan C = \frac{c}{a}$ or $c = a \tan C$. When we know a and c, b, the hypotenuse can be found by the fact $b^2 = a^2 + c^2$ or by using the trigonometrical ratio $\sec C = \frac{b}{a}$ or $b = a \sec C$.

There are ready made tables of Trigonometrical Ratios (see p. 186-188), by referring to which we can find the value of any trigonometrical ratio for any angle. These ratios are expressed as decimals correct to four decimals. The ratios of 30° or $\frac{\pi}{6}$, 45° or $\frac{\pi}{4}$ and 60° or $\frac{\pi}{3}$ are already known to you (see chapter 4).

Example 1: Solve the triangle ABC given that $B=90°$, $a=5$ cm, $b=10$ cm.

By Pythagoras' theorem

$$c=\sqrt{b^2-a^2}=\sqrt{100-25} \text{ cm.}$$
$$=\sqrt{75} \text{ cm}=5\sqrt{3} \text{ cm}=5\times 1{\cdot}73 \text{ cm}=8{\cdot}66 \text{ cm.}$$

(approximately)

$\sin A=\frac{5}{10}=\frac{1}{2}$, so $A=30°$.

Hence $C=90°-30°=60°$.

You should note that the solution of the equation $\sin A=\frac{1}{2}$ is $A=30°$, $A=150°$, $A=390°$, etc. but here the angle A is less than 90°, hence the only admissible solution of $\sin A=\frac{1}{2}$ is $A=30°$.

Example 2: Given $A=90°$, $B=60°$, $b=5$ cm, solve the triangle ABC.

$$C = 90°-60°=30°$$
$$\frac{5}{a} = \sin 60°$$
$$\therefore a=\frac{5}{\sin 60°}=\frac{10}{\sqrt{3}}=\frac{10\sqrt{3}}{3}, \text{ as } \sin 60° = \frac{\sqrt{3}}{2}$$
$$\frac{c}{5} = \tan 30°$$
$$\therefore c=5 \tan 30° = \frac{5}{\sqrt{3}}=\frac{5\sqrt{3}}{3}=2{\cdot}89 \text{ cm.}$$

Example 3: Given $C=90°$, $c=50$ cm, $B=20°$, solve the triangle.

Here $A=90°-020°=7°$

$$\frac{b}{50}=\sin 20°$$
$$b=50 \sin 20°=50\,({\cdot}3420) \text{ from the trigonometric tables.}$$
$$= 17{\cdot}10$$

Also $\frac{a}{50}=\cos 20°={\cdot}9307$ $\therefore a=50\times {\cdot}9397$ or $46{\cdot}985$ cm.

3. Problems on heights and distances.

In problems on heights and distances, the following terms are frequently used in referring to angles in a vertical plane.

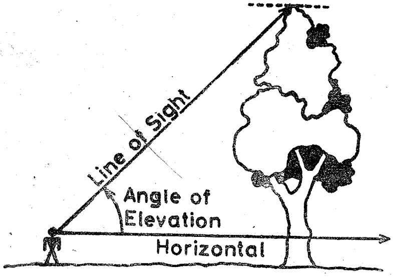


Fig. 10.2

(a) *The angle of elevation.* If the top of a tree or any object observed is above the horizontal, the angle determined by the horizontal and the line of sight is called the angle of elevation.

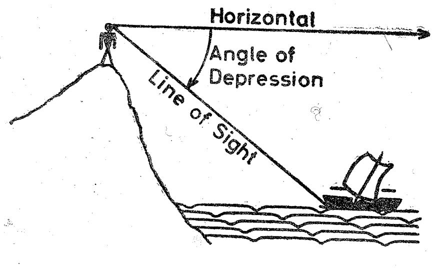


Fig. 10.3

(b) Here a boat is sailing on sea and a man sees it from the top of a hill overlooking the sea. In such cases, the object observed is below the horizontal. The angle determined by the horizontal and the line of sight is called the *angle of depression.*

Example 4: The angle of elevation of a tower at a point 150 m from its base is 25°. Find the height of the tower.

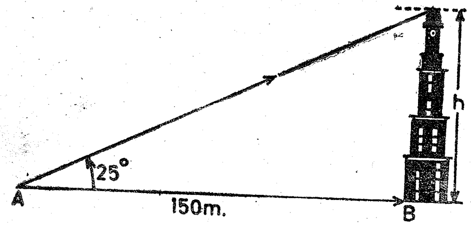


Fig. 10.4

$$\frac{h}{150}=\tan 25°$$
$$\therefore h = {\cdot}4663\times 150 \text{ m}$$
$$= 69{\cdot}9450 \text{ m}$$

$\therefore$ height of the tower is approximately $69{\cdot}9$ m.

Example 5: At a certain place it was found that at mid-day on 22nd December, the shadow cast by a pole 11·5 m high was 10 m. Find the sun's altitude at the place.

The sun's altitude is θ in degrees, where $\tan\theta = \frac{11{\cdot}5}{10} = 1{\cdot}15$. By referring to the trigonometric table, we see that $\theta = 49°$ approximately.

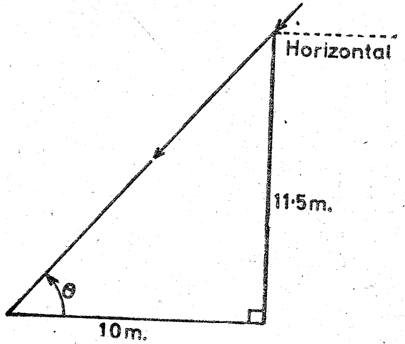


Fig. 10.5

Example 6: A person walking along a straight road towards a hill, observes that from two points A and B on the road one km. apart, the angles of elevation of the hill are 30° and 45°. Find the height of the hill.

In the figure, PO is the height of the hill. Let x metres be the distance BO and $AB = 1000$ m.

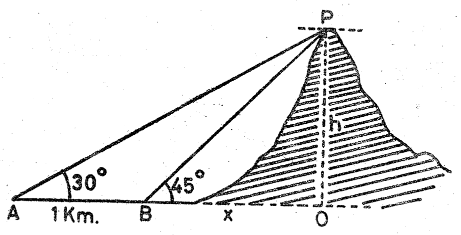


Fig. 10.6

$$\therefore \tan 30° = \frac{h}{1000+x} \text{ and } \tan 45° = \frac{h}{x} = 1$$

Subba Rao,
Telegraphs.

$\therefore h = x$ and

$\cdot 5774(1000 + x) = x$ (by referring to the table, we see that $\tan 30° = \cdot 5774$)

$\therefore 577{\cdot}4 + x(\cdot 5774) = x$

or $x(1 - \cdot 5774) = 577{\cdot}4$

$\therefore \cdot 4226x = 577{\cdot}4$

$$x = h = \frac{577{\cdot}4}{\cdot 4226} = \frac{5774000}{4226} \text{ m}$$

$$= 1366{\cdot}3 \text{ m.}$$

Exercise 1

1. A Surveyor notices a stone at a point P on the opposite bank of a river directly in front of A at which he is standing on the other bank. Then he walks along the bank to B at a distance of 150 m. and notices that the stone on the other side makes an angle of 27° with BA. Find the width of the river.

2. Find the angle of elevation of the sun when the shadow of a pole 3 metres high is $3\sqrt{3}$ m long.

3. A ladder 8 metres long just reaches the top of a wall and makes an angle of 30° with the vertical through the top of the ladder. Find the height of the wall and the distance of the foot of the ladder from the wall.

4. Two poles are 10 metres and 20 metres high. The line joining their tops makes an angle of 15° with the horizontal. Find the distance between the poles.

5. A flagstaff stands on the top of a pole. At a distance of 15 metres, the angles of elevation of the tops of the pole and the flagstaff are 30° and 60°. Find the length of the flagstaff.

6. From the foot of a tower, the angle of elevation of the top of a monument is 60°, and from the top of the tower, the angle of elevation is 30°. If the tower is 14 metres high, find the height of the monument.

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7. From the top of a building 25 metres high, the angles of depression of two objects on the horizontal ground in line with the building are 45° and 30°. Find the distance between the objects.

8. A building is 6 m high. When rainfalls at an angle of 3° with the vertical, for what horizontal distance from the foot of the wall will the ground remain dry? 6·f

9. Two spires of heights 14 metres and 20 metres respectively stand on level round. The angles of elevation of their tops from a point on the line joining their feet are respectively 45° and 60°. Find the distance between the spires.

10. From the top of a hill 100 metres high, the angles of depression of the top and bottom of a vertical column are 45° and 60° respectively. Find the height of the column.

11. The horizontal distance between two towers is 27 metres and the angular depression of the top of the first as seen from the top of the second is 30°. The angular elevation of the top of the second as seen from the bottom of the first is 60°. Find the heights of the towers.

12. A bridge over a railroad track is 12·50 m high. If th approach to the bridge has an angular elevation of 15°, find th length of the approach measured along the road surface.

13. A person walking due *E* observes two objects both in th N.E. direction. After walking 200 metres, one of the objects due N of him, and the other is N.W. Find the distance betwee the objects.

14. A ship sailing due E. observes a light-house bearing N. E at a distance of 20 kilometres. After sailing for two hours, th light-house bears N.W. Find the speed of the ship.

15. Two ships sail from a harbour simultaneously in directio N.W. and N.E. at 12 and 15 kilometres per hour respectivel Find the direction of the second ship as observed from the fir after 2 hours.

16. From the roof of a house 10 metres high, the angle of elev tion of the top of a tree is 20° and the angle of depression of its fo is 40°. Fi [illegible] f the tree and its distance from the hous

17. Two pillars of equal height stand on either side of a road-y which is 100 metres wide; at a point in the roadway between e pillars the elevations of the tops of the pillars are 60° and 30°. nd their height and the position of the point.

18. The angle of elevation of the top of a tower is observed be 60°; at a point 40 m above the first point of observation the vation is found to be 45°; find the height of the tower and its rizontal distance from the points of observation.

19. The shadow of a tower standing on a level plane is found to 60 metres longer when the sun's altitude is 30° than when it is . Prove that the height of the tower is $30(1+\sqrt{3})$ m.

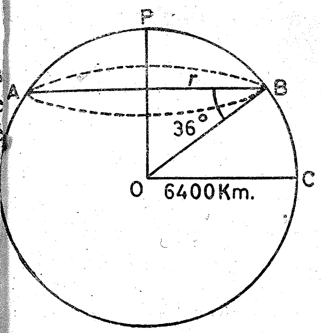

Fig. 10.7

20. Considering the earth to be a sphere of radius 6400 km, find the radius of the 36th parallel of latitude.

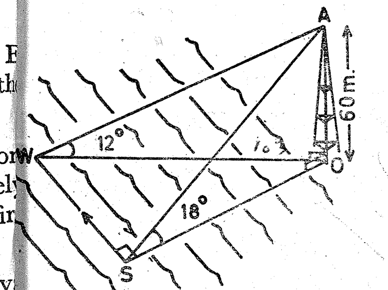

Fig. 10.8

21. From the top of a tower, 60 m above the water, the angle of depression of a boat due south is 18°. Calculate the speed in km p.h. of the boat if, after it moves due west for 3 minutes, the angle of depression is 12°.

4. The Sine Law for a triangle.

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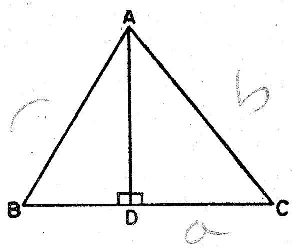

Fig. 10.9

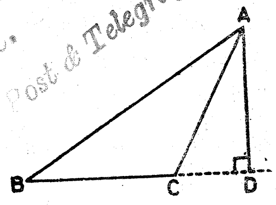

Fig. 10.10

First let us consider a triangle in which all the three angles A, B, C are acute angles. [See fig. 9]. It is usual to denote the lengths of opposite sides as a, b, c so that $BC=a$, $CA=b$ and $AB=c$.

Draw $AD \perp BC$

Now $\sin B=\dfrac{AD}{AB}$ and $\sin C=\dfrac{AD}{AC}$

$$\therefore \sin B : \sin C=\frac{AD}{AB} : \frac{AD}{AC}=\frac{AD}{AB}\times\frac{AC}{AD}=\frac{AC}{AB}$$
$$=AC : AB$$
$$=b : c$$

By drawing a perpendicular from B to the side CA, we can similarly show that

$\sin C : \sin A=c : a$

$\therefore \sin A : \sin B : \sin C=a : b : c$

We can also write the relation as

$\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$. This statement is called the Sine Law.

This can be proved even when one angle is obtuse.

In the second figure

$\sin B=\dfrac{AD}{AB}$ and $\sin C=\dfrac{AD}{AC}$

$$\therefore \frac{\sin B}{\sin C}=\frac{AD}{AB}\times\frac{AC}{AD}=\frac{AC}{AB}=\frac{b}{c}$$

In a similar manner $\dfrac{\sin A}{\sin B}=\dfrac{a}{b}$

$\therefore \dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$ holds good even in an obtuse-angled triangle.

Example 7: Solve the triangle in which $a=4{\cdot}5$ cm, $B=64°$ and $C=47°$.

The Sine Law is used to solve a triangle when two angles and one side of the triangle are given. Since the trigonometric tables are given for degrees only, it is not possible with these tables to calculate the lengths accurately, when angles given include minutes and seconds. The fractions of a degree of an angle have to be ignored in the circumstances.

A is immediately found as the supplement of $B+C$

i.e., $180°-(64°+47°)=180°-111°=69°$

$$\frac{a}{\text{Sin } A}=\frac{b}{{\cdot}8988}=\frac{4{\cdot}5}{{\cdot}9336} \quad \therefore b=4{\cdot}5\times\frac{8988}{9336}$$

$$\log b = \log 4{\cdot}5+\log 8988-\log 9336$$
$$= 0{\cdot}6532+3{\cdot}9537-3{\cdot}9700$$
$$= 0{\cdot}6369$$

Using anti-logarithm table, $b=4{\cdot}334$ or $4{\cdot}3$ cm roughly.

$$\frac{c}{\sin C}=\frac{c}{\sin 47°}=\frac{c}{{\cdot}7314}=\frac{a}{\sin A}=\frac{4{\cdot}5}{{\cdot}9336}$$

$$\therefore c=\frac{{\cdot}7314\times4{\cdot}5}{{\cdot}9336} \quad \text{or } 4{\cdot}5\times7314\div9336$$

$$\log c = \log 4{\cdot}5 + \log 7314 - \log 9336$$
$$= 0{\cdot}6532 + 3{\cdot}8641 - 3{\cdot}9700$$
$$= 0{\cdot}5473$$
$$c = 3{\cdot}527 \quad \therefore c = 3{\cdot}5 \text{ cm roughly.}$$

5. The Cosine Law.

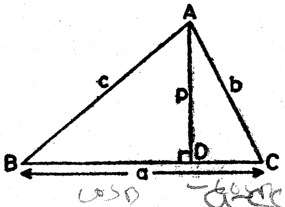

Fig. 10.11

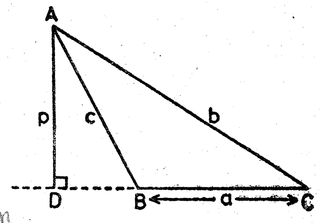

Fig. 10.12

Let *ABC* be any triangle. In the first figure the angles
acute and in the second figure angle *B* is obtuse. *AD* ⊥
In the first figure, *D* is *between B* and *C* and in the second fig
D is on *CB* produced.

In the first figure $BD = c \cos B$ and AD or $p = c \sin B$

„ second „ $BD = -c \cos B$ and „ „ „

in either case $DC = a - c \cos B$.

By the theorem of Pythagoras,

$$\begin{aligned} b^2 &= p^2 + DC^2 \\ &= c^2 \sin^2 B + (a - c \cos B)^2 \\ &= c^2 \sin^2 B + a^2 + c^2 \cos^2 B - 2\, ac \cos B \\ &= c^2 + a^2 - 2ac \cos B \end{aligned}$$

$$\therefore 2ac \cos B = a^2 + c^2 - b^2 \text{ or } \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

Similarly $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ and $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$.

These results are usually known as the Cosine Law and elp us in solving triangles. The student will remember the xtensions of Pythagoras' Theorem in Plane Geometry viz. any triangle, the squares on the side opposite an acute ngle is less than the sum of the squares on the sides contain- g the acute angle, by twice the rectangle contained by ither of these sides and the projection on it of the other side djacent to the acute angle.

This and the other theorem of an obtuse-angled triangle are he same as the Cosine Law.

Example 8: Find the angles of a triangle whose sides are 3 cm, 14 cm, and 15 cm.

$$\begin{aligned} \cos B &= \frac{a^2 + c^2 - b^2}{2ac} \\ &= \frac{14^2 + 15^2 - 13^2}{2 \cdot 14 \cdot 15} \\ &= \frac{196 + 225 - 169}{420} = \frac{252}{420} \\ &= \tfrac{3}{5} = {\cdot}60 \end{aligned}$$

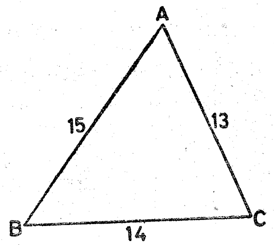

Fig. 10.13

Looking at the Trigonometric tables, we see that the nearest angle in degrees whose cosine is ·60 is 53°.

$$\cos C = \frac{13^2 + 14^2 - 15^2}{2{\cdot}13{\cdot}14} = \frac{169 + 196 - 225}{364} = \frac{140}{364}$$
$$= \tfrac{5}{13} = {\cdot}3846$$

The nearest angle whose cosine is ·3846 is 67°.

The other angle *A* is, ∴ 60°.

∴ $A = 60°$, $B = 53°$ and $C = 67°$.

6. The area of a triangle in terms of the lengths of the sides.

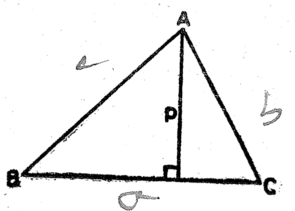

Fig. 10.14

The area $= \frac{1}{2}\,BC\cdot p = \frac{1}{2}a\cdot p$ where $p = c \sin B$.

$$= \tfrac{1}{2}ac \sin B$$

$$= \tfrac{1}{2}ac\sqrt{1-\cos^2 B}$$

$$= \tfrac{1}{2}ac\sqrt{1-\left(\frac{a^2+c^2-b^2}{2ac}\right)^2}$$

$$= \tfrac{1}{2}ac\frac{\sqrt{4a^2c^2-(a^2+c^2-b^2)^2}}{2ac}$$

$$= \tfrac{1}{4}\sqrt{(2ac+a^2+c^2-b^2)\,(2ac-a^2-c^2+b^2)}$$

$$= \tfrac{1}{4}\sqrt{\{(a+c)^2-b^2\}\,\{b^2-(a-c)^2\}}$$

$$= \tfrac{1}{4}\sqrt{(a+b+c)\,(a+c-b)\,(a+b-c)\,(b+c-a)}$$

$$= \sqrt{s(s-a)\,(s-b)\,(s-c)}$$

where $2s = a+b+c$ so that $a+c-b = 2(s-b)$ etc. This is known as Heron's formula for the area of a triangle.

7. Ambiguous and impossible cases.

When two sides and an angle (other than the included angle) are given, there may be two triangles, one triangle or no triangle according to the data. You have learnt this in Geometry. Let us see this in Trigonometry as well.

area [illegible] be prove [illegible] three sides

5-8-83 $\frac{1}{2}$

Example 7: Find the other angles and the sides of the triangle in which $a = 20$ cm, $b = 30$ cm and $A = 30°$.

$$\frac{20}{\sin 30°} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$\therefore 40 = \dfrac{30}{\sin B}$ or $\sin B = \frac{3}{4} = \cdot 75$ since $\sin 30° = \frac{1}{2}$.

Since $\sin B = \sin(180° - B)$, there may be two values of B. The acute angle is roughly 49° (by referring to the trigonometric tables) and the obtuse angle is 131°. C is either 101° or 19°.

Now $\dfrac{c}{\sin C} = \dfrac{a}{\sin A} = 40$

$\therefore c = 40 \sin 101° = 40 \sin 79° = 40 \times \cdot 9816$
$= 39\cdot 264 = 39\cdot 2$ cm roughly.

c is also equal to $40 \sin 19° = 40 \times \cdot 3256$
$= 13\cdot 024$ or 13 cm roughly.

$\therefore a = 20$ cm, $b = 30$ cm, and $c = 39\cdot 2$ or 13 cm.

Both these solutions are valid, since the sum of the lengths of any two sides is greater than the length of the third side. Similarly the sum of the three angles in either case is not greater than 180°. In some cases this may not hold good and then there will be no solution.

Exercise 2

1. Write down an expression for each of the lengths of sides a, b, c in the following figures.

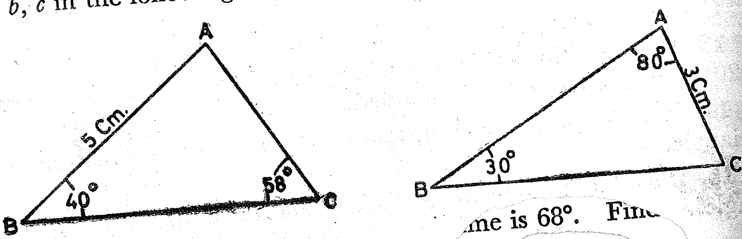

Fig. 10.15

me is 68°. Fin

d the distance

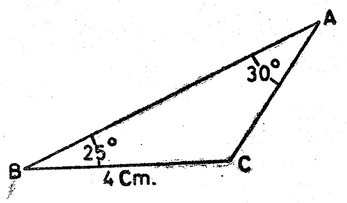

Fig. 10.17

Using the trigonometric tables where necessary, solve the following triangles, stating angles to the nearest degree.

2. $a=6$ cm, $B=32°$ and $C=120°$.

3. $a=4{\cdot}2$ cm, $A=78°$ and $B=40°$.

4. $b=8{\cdot}4$ cm, $A=63°$ and $C=42°$.

5. $c=10$ cm, $A=70°$ and $B=30°$.

6. $b=7$ cm, $A=55°$ and $C=45°$.

7. $a=6$ cm, $B=22°$ and $C=138°$.

8. $c=4{\cdot}8$ cm, $A=40°$ and $C=100°$

9. $b=10$ cm, $A=35°$ and $B=60°$.

10. In an isosceles triangle, the length of each of the equal sides is 10 cm, and each of the equal angles has a measurement of 68°. Determine the length of the third side.

11. Two adjacent sides of a parallelogram make angles 42° and 68° respectively, with the diagonal. If the length of the diagonal is 10 cm, find the lengths of the sides of the parallelogram and the other diagonal.

12. From a town A the road goes straight north 4·5 km, then bears N 68° E to a town B which is N 35° E of A. Find the distance between A and B by airline and by road.

Use the Cosine Law for the following.

13. Find the third side of a triangle having 10 cm, 15 cm and 120° as two sides and the included.

14. If $a=2{\cdot}5$ cm, $b=2$ cm, and $c=$ 3 m, find the angles of the triangle.

15. If $b=8$ cm, $c=6$ cm, $\angle A=70°$, find the other side and the angles.

16. If $a=4$ cm, $c=5$ cm, $\angle B=72°$, find the other side and the angle.

17. If $a=12$ cm, $b=11$ cm, and $c=12{\cdot}5$ cm, find the three angles of the triangle.

18. Two sides and the included angle of a parallelogram are 12 cm, 15 cm and 105°. Find the lengths of the diagonals and the angles made by them.

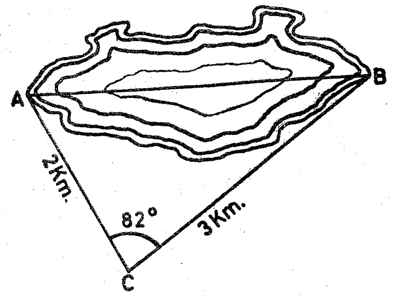

Fig. 10.18

19. To find the length of a pond, the following measurements are made: $AC=2$ km, $BC=3$ km, and angle $C=82°$. Find the length of the pond to the nearest metre.

20. A parallelogram has 12 cm, and 15 cm. The longer diagonal is 21 cm. Find the angles of the parallelogram.

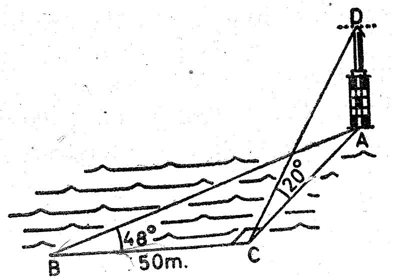

Fig. 10.19

21. In the adjoining, plane ABC is horizontal and the plane ADC is vertical. To calculate the height of a tower situated on the opposite side of a river, the surveyors made the measurements as under:

$A\hat{C}D=20°$ and $A\hat{B}C=48°$ and $BC=50$ m.

Determine the length of the tower.

22. When an air-craft is directly above a point O on the ground, its angle of elevation from a point A due south of O is 72°. From another point B which is 50 metres due west of O, the angle of elevation of the air-craft at the same time is 68°. Find the height of the air-craft and the distance OA.

CHAPTER 11

Addition and Multiple Angle Formulae

1. Addition formulae.

We first prove the formula

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$.

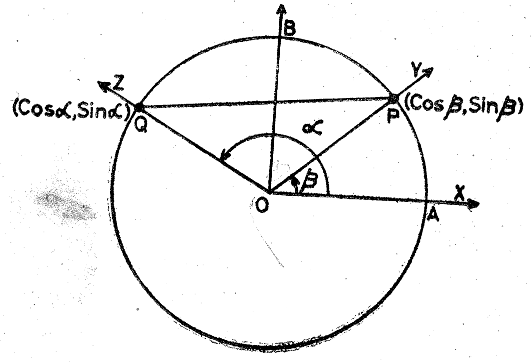

Fig. 11.1

Let OX be the initial ray and OY, OZ be the terminal rays of angles β and α. Draw a unit circle about the origin, intersecting the rays OX, OY, OZ, at A, P, Q respectively. Thus $OA=OP=OQ=$unit. Taking OA and OB as x-axis and y-axis, the coordinates of P and Q are $(\cos\beta, \sin\beta)$ and $(\cos\alpha, \sin\alpha)$, whatever be the measurements of the angles.

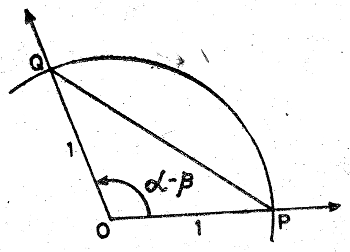

Fig. 11.2

$\therefore$ the distance $PQ=\sqrt{(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2}$

$\therefore PQ^2=\cos^2\alpha+\sin^2\alpha+\cos^2\beta+\sin^2\beta-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta$

$=2-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta \quad \ldots.(1)$

Now according to the Cosine Law, considering $\triangle OPQ$, we have $PQ^2=OP^2+OQ^2-2OP\cdot OQ\cos(\alpha-\beta)$

$$=1+1-2\cos(\alpha-\beta)$$

$$=2-2\cos(\alpha-\beta) \quad \ldots.(2)$$

Comparing (1) and (2) above, we have

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta \quad \ldots.\ (I)$$

This formula holds good for any angles positive or negative.

If we write β' for $-\beta$, we have, $\cos(\alpha+\beta')=\cos\alpha\cos\beta'-\sin\alpha\sin\beta'$ as $\cos\beta=\cos\beta'$, and $\sin\beta=-\sin\beta'$. If we drop the dashes we have $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$ $\ldots\ldots$ (II)

$$\text{Now }\sin(\alpha+\beta)=\cos\left(\frac{\pi}{2}-\overline{\alpha+\beta}\right)$$

$$=\cos\left(\overline{\frac{\pi}{2}-\alpha}-\beta\right)$$

$$=\cos\left(\frac{\pi}{2}-\alpha\right)\cos\beta+\sin\left(\frac{\pi}{2}-\alpha\right)\sin\beta$$

$$=\sin\alpha\cos\beta+\cos\alpha\sin\beta \quad \ldots\ldots(III)$$

If we write $-\beta$ instead of β, we have

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta \quad \ldots.(IV)$

as $\sin(-\beta)=-\sin(\beta)$ and $\cos(-\beta)=\cos\beta$.

Dividing III by II

$$\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\tan(\alpha+\beta)=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$

$$=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\text{ dividing}$$

both the numerator and deno[illegible] $\cos\beta$.

$$\therefore \tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \text{ in terms } \tan\alpha \text{ and } \tan\beta \qquad \ldots\ldots\ldots \text{(V)}$$

$$\text{Also } \tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} \qquad \ldots\ldots\ldots \text{(VI)}$$

Putting $-\beta$ in place of β in (V).

(V) is valid only if $\tan\alpha\tan\beta\neq 1$ and (VI) is valid if $\tan\alpha\tan\beta\neq -1$.

These are six basic addition formulae, which are worth remembering.

Example 1: Prove that $\cot(\alpha+\beta)=\dfrac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$

$$\cot(\alpha+\beta)=\frac{1}{\tan(\alpha+\beta)}=\frac{1-\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$$

$$=\frac{1-\dfrac{1}{\cot\alpha\cot\beta}}{\dfrac{1}{\cot\alpha}+\dfrac{1}{\cot\beta}}$$

$$=\frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta} \text{ multiplying the above result by } \cot\alpha\cot\beta.$$

Here it is assumed that $\cot\alpha+\cot\beta\neq 0$, $\sin\alpha\neq 0$ and $\sin\beta\neq 0$.

Example 2: Show that $\cos 240^\circ+\cos 45^\circ=-\frac{1}{2}+\dfrac{1}{\sqrt{2}}$

$$\cos 240^\circ=\cos(180^\circ+60^\circ)=\cos 180^\circ\cos 60^\circ-\sin 180^\circ\sin 60^\circ$$
$$=-\cos 60^\circ-0\times\sin 60^\circ$$
$$=-\tfrac{1}{2}$$

$$\therefore \cos 240^\circ+\cos 45^\circ=-\tfrac{1}{2}+\frac{1}{\sqrt{2}}.$$

Even otherwise, we can easily see the result, as

$\cos (1$ [illegible] $-\cos 60^\circ$.

Exercise 1

1. By using the formulae for $\sin(\alpha-\beta)$ and $\cos(\alpha-\beta)$, show that $\sin\left(\dfrac{\pi}{2}-\theta\right)=\cos\theta$ and $\cos\left(\dfrac{\pi}{2}-\theta\right)=\sin\theta$, for any value of θ.

2. By using the addition formulae, show that $\sin(-\theta)=-\sin\theta$, $\cos(-\theta)=\cos\theta$ and $\tan(-\theta)=-\tan\theta$.

3. Assuming the formulae for $\cos(\alpha-\beta)$ and $\sin(\alpha-\beta)$, prove that $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$ and

$$\cot(\alpha-\beta)=\frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}.$$

4. By using the formulae for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ show that $\sin\left(\dfrac{\pi}{2}+\theta\right)=\cos\theta$ and $\cos\left(\dfrac{\pi}{2}+\theta\right)=-\sin\theta$, $\cos(\pi+\theta)=-\cos\theta$ and $\sin(\pi+\theta)=-\sin\theta$.

5. Prove that $\cos 15^\circ=\cos(45^\circ-30^\circ)=\dfrac{\sqrt{3}+1}{2\sqrt{2}}$,

$\cos 75^\circ=\cos(45^\circ+30^\circ)=\dfrac{\sqrt{3}-1}{2\sqrt{2}}$ and $\sin 75^\circ=\dfrac{\sqrt{3}+1}{2\sqrt{2}}$.

6. Find $\sin(\alpha+\beta)$, $\sin(\alpha-\beta)$, $\cos(\alpha-\beta)$ and $\cos(\alpha+\beta)$, if $\sin\alpha=\frac{3}{5}$, $\sin\beta=\frac{5}{13}$, where α is an acute and β is an obtuse angle.

7. Prove that $\sin(240^\circ+45^\circ)=-\dfrac{(\sqrt{3}+1)}{2\sqrt{2}}$ and

$\cos(240^\circ+45^\circ)=\dfrac{\sqrt{3}-1}{2\sqrt{2}}$.

8. Without the use of tables, find
(i) $\sin 165^\circ$, (ii) $\cos(-15)^\circ$, (iii) $\tan 195^\circ$.

Prove the following results:

9. $\sin 2\theta\cos\theta-\cos 2\theta\sin\theta=\sin\theta$.

10. $\sin(30^\circ+\theta)+\cos(60^\circ+\theta)=\cos\theta$.

11. $\cos\left(\dfrac{\pi}{4}-\theta\right)=\sin\left(\dfrac{\pi}{4}+\theta\right)$.

12. $\cos \frac{3}{4}\theta \cos \frac{\theta}{4} - \sin \frac{3\theta}{4} \sin \frac{\theta}{4} = \cos \theta.$ ✓

13. $\sin(\alpha+\beta)\sin(\alpha-\beta) = \sin^2\alpha - \sin^2\beta = \cos^2\beta - \cos^2\alpha.$

14. $\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta = \cos^2\beta - \sin^2\alpha.$

15. $\dfrac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha+\beta)+\cos(\alpha-\beta)} = \tan\alpha.$

16. $\cos\beta\cos(\beta-\alpha) + \sin\beta\sin(\beta-\alpha) = \cos\alpha.$

17. $\dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \dfrac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta}.$

18. $\dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} = \tan\alpha + \tan\beta.$

19. $\cos(\alpha+\theta)\cos(\beta+\theta) + \sin(\alpha+\theta)\sin(\beta+\theta)$
$= \cos\alpha\cos\beta + \sin\alpha\sin\beta.$

20. $\sin(\alpha+\beta) + \cos(\alpha-\beta) = (\cos\alpha + \sin\alpha)(\cos\beta + \sin\beta).$

2. Multiple Angle Formulae.

$$\sin 2\theta = \sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta \quad \ldots\ldots(\text{VII})$$

$$\cos 2\theta = \cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$$
$$\text{or } 2\cos^2\theta - 1 \text{ or } 1 - 2\sin^2\theta \quad \ldots\ldots(\text{VIII})$$

$$\begin{aligned}
\cos 3\theta = \cos(2\theta+\theta) &= \cos 2\theta\cos\theta - \sin\theta\sin 2\theta \\
&= (\cos^2\theta - \sin^2\theta)\cos\theta - \sin\theta\,(2\sin\theta\cos\theta) \\
&= \cos^3\theta - \sin^2\theta\cos\theta - 2\sin^2\theta\cos\theta \\
&= \cos^3\theta - 3(1-\cos^2\theta)\cos\theta \\
&= 4\cos^3\theta - 3\cos\theta \quad \ldots\ldots(\text{IX})
\end{aligned}$$

$$\begin{aligned}
\sin 3\theta = \sin(2\theta+\theta) &= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta \\
&= 2\sin\theta\cos^2\theta + (\cos^2\theta - \sin^2\theta)\sin\theta \\
&= 2\sin\theta(1-\sin^2\theta) + (1-2\sin^2\theta)\sin\theta \\
&= 3\sin\theta - 4\sin^3\theta \quad \ldots\ldots(\text{X})
\end{aligned}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} = \frac{2\sin\theta\cos\theta}{\cos^2\theta} \div \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta}$$
$$= 2\tan\theta \div (1-\tan^3\theta) = \frac{2\tan\theta}{1-\tan^2\theta} \quad \ldots.(\text{XI})$$

In a similar manner

$$\begin{aligned}
\sin 4\theta &= 2\sin 2\theta\cos 2\theta = 4\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta) \\
&= 4\sin\theta\cos^3\theta - 4\cos\theta\sin^3\theta \\
\cos 4\theta &= \cos^2 2\theta - \sin^2 2\theta = (2\cos^2 2\theta - 1) \\
&= 2(2\cos^2\theta - 1)^2 - 1 \\
&= 8\cos^4\theta - 8\cos^2\theta + 1.
\end{aligned}$$

Example 3: Find $\sin\frac{\pi}{2}$, $\cos\frac{\pi}{2}$, assuming $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

$$\sin\frac{\pi}{2} = 2\sin\frac{\pi}{4}\cos\frac{\pi}{4} = 2\times\frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{2}} = 1$$

$$\cos\frac{\pi}{2} = \cos^2\frac{\pi}{4} - \sin^2\frac{\pi}{4} = \tfrac{1}{2} - \tfrac{1}{2} = 0.$$

Example 4: Prove that

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}, \quad \cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\text{and } \tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}, \quad 1+\cos\theta \neq 0.$$

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2}$$

$$\therefore 2\cos^2\frac{\theta}{2} = 1 + \cos\theta \qquad \therefore \cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\text{Also } 2\sin^2\frac{\theta}{2} = 1 - \cos\theta \qquad \therefore \sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}.$$

3. Formulae for sin A+sin B, cos A+cos B etc.

We know $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$ and $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$.

Adding the two, $\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$.

If we put $A=\alpha+\beta$ and

$B=\alpha-\beta$, we have $\alpha=\dfrac{A+B}{2}$ and $\beta=\dfrac{A-B}{2}$

$$\therefore \sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2} \qquad \text{......(XII)}$$

Also
$$\begin{aligned}\sin A-\sin B&=\sin(\alpha+\beta)-\sin(\alpha-\beta)\\&=2\cos\alpha\sin\beta\\&=2\cos\frac{A+B}{2}\sin\frac{A-B}{2} \qquad \text{......(XIII)}\end{aligned}$$

Similarly $\cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos\alpha\cos\beta$

and $\cos(\alpha+\beta)-\cos(\alpha-\beta)=-2\sin\alpha\sin\beta$.

$$\therefore \cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2} \qquad \text{....(XIV)}$$

and
$$\begin{aligned}\cos A-\cos B&=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}\\ \text{or } &2\sin\frac{A+B}{2}\sin\frac{B-A}{2} \qquad \text{......(XV)}\end{aligned}$$

Example 5: Prove that $\dfrac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta}=\tan\dfrac{\alpha+\beta}{2}\cot\dfrac{\alpha-\beta}{2}$

$$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \qquad \text{....(1)}$$

$$\sin\alpha-\sin\beta=2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2} \qquad \text{....(2)}$$

Dividing (1) by (2), we have
$$\frac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta}=\frac{\tan\dfrac{\alpha+\beta}{2}}{\tan\dfrac{\alpha-\beta}{2}}$$
$$=\tan\frac{\alpha+\beta}{2}\cot\frac{\alpha-\beta}{2}.$$

Exercise 2

1. Assuming the values of $\sin\frac{\pi}{6}$ and $\cos\frac{\pi}{6}$, work out $\sin\frac{\pi}{3}$, $\cos\frac{\pi}{3}$ and $\tan\frac{\pi}{3}$.

2. If $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$ and $\cos\frac{\pi}{3}=\frac{1}{2}$, work out $\sin\frac{2\pi}{3}$, $\cos\frac{2\pi}{3}$ and $\tan\frac{2\pi}{3}$.

3. Assuming the values of sin 30° and cos 30°, find the values of sin 90°, cos 90°, sin 180° and cos 180°.

4. Assuming the values of sin 90° and cos 90°, find the values sin 45° and cos 45°.

5. Assuming $\cos\frac{\pi}{3}=\frac{1}{2}$, work out $\sin\frac{\pi}{6}$, $\cos\frac{\pi}{6}$ and $\tan\frac{\pi}{6}$ with the proper signs.

6. If $\sin\theta=\frac{4}{5}$, find $\sin 2\theta$, $\cos 2\theta$, and $\sin 3\theta$.

Prove the following identities:

7. $\cos 2\theta=(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)$.

8. $\sin 4\theta=4\sin\theta\cos\theta(\cos^2\theta-\sin^2\theta)$.

9. $\tan\theta+\cot\theta=2\operatorname{cosec}2\theta$.

10. $\dfrac{\sin 2\theta+\sin\theta}{\cos 2\theta+\cos\theta+1}=\tan\theta$.

11. $\sin 2\theta=2\sin^3\theta\cos\theta+2\cos^2\theta\sin\theta$.

12. $\dfrac{\sin 3\theta}{\sin\theta}-\dfrac{\cos 3\theta}{\cos\theta}=2$.

13. $\frac{1}{2}\sin 2\theta=\frac{\tan\theta}{1+\tan^2\theta}$.

14. $\cos 2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}$.

15. $\tan 2\theta=\frac{2}{\cot\theta-\tan\theta}$

16. $2\cos\theta\operatorname{cosec}2\theta=\operatorname{cosec}\theta$.

17. $\frac{\cos^3\theta-\sin^3\theta}{\cos\theta-\sin\theta}=1+\frac{1}{2}\sin 2\theta$. — gnf

18. $\frac{\sin 3\theta}{\sin\theta}-\frac{\cos 3\theta}{\cos\theta}=4\cos 2\theta$.

19. $\tan\theta=\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$.

20. $\cot\frac{\theta}{2}-\tan\frac{\theta}{2}=2\cot\theta$.

21. $\tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}$. Use this result to find tan $(22\frac{1}{2})°$, taking $\theta=45°$. ng

22. $\cos 6\theta+\cos 4\theta=2\cos\theta\cos 5\theta$.

23. $\sin 4A+\sin 4B=2\sin(A+B)\{\cos(3A-B)+\cos(3B-A)\}$

24. If $A+B+C=180°$, prove that

$$\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

gd is very gnf

CHAPTER 12

Simple Trigonometric Equations and Graphs

1. In chapter 5, we have seen that the results of the type $\sin^2\theta+\cos^2\theta=1$ are identities, which are satisfied for all values of θ. On the other hand, equations of the type $\sin\theta=1$ are not identities, because, such equations are not satisfied for all values of θ; for instance, the above equation is true for $\theta=\frac{\pi}{2}$ but not for $\theta=\frac{\pi}{3}$. Equations involving trigonometric functions, which are not identities are called *trigonometric equations*. In general, they will possess an infinity of solutions. For instance, the equation $\sin\theta=1$ is satisfied for $\theta=\frac{\pi}{2}$, $\theta=2\pi+\frac{\pi}{2}, 4\pi+\frac{\pi}{2}, 6\pi+\frac{\pi}{2}\ldots\ldots$; because $\sin\theta$ has the same value when θ is increased by 2π, 4π, etc. In what follows we shall find out solutions for θ which lie within 0 and 2π, that is, those values of θ which satisfy $0\leqslant\theta<2\pi$. These are the only solutions which have practical use. To get the general solution we can add 2π, 4π, 6π and so on to the solutions obtained if the equations are in sine and cosine and π, 2π, 3π, ... if they are in tangent and cotangent. In general, if $\sin\theta=\sin\alpha$, then $\theta=\alpha$ or $\pi-\alpha$, if $\cos\theta=\cos\alpha$, $\theta=\alpha$ or $2\pi-\alpha$ and if $\tan\theta=\tan\alpha$, $\theta=\alpha$ or $\pi+\alpha$.

Example 1: Solve $2\tan^2\theta+\sec^2\theta=2$.

We have $2\tan^2\theta+1+\tan^2\theta=2$

$\therefore 3\tan^2\theta=1$

$\therefore \tan\theta=\pm\frac{1}{\sqrt{3}}$

From $\tan\theta=+\frac{1}{\sqrt{3}}$ we get $\theta=\frac{\pi}{6}$, $\pi+\frac{\pi}{6}$ and

from $\tan\theta=-\frac{1}{\sqrt{3}}$ we get $\theta=\frac{5\pi}{6}$, $\pi+\frac{5\pi}{6}$.

After substituting these values of θ in the equation, we see that the equation is satisfied by all these values.

Hence the solutions are $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

Checking must be done every time, because it may happen that some values which are obtained by algebraic operations may not satisfy the equation concerned. We shall illustrate this by an example.

Example 2: Solve $\sec\theta+\tan\theta=0$

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Now $\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}=0$

i.e., $\frac{1+\sin\theta}{\cos\theta}=0$

Multiplying by $\cos\theta$ we have,

$1+\sin\theta=0$. Hence $\theta=\frac{3\pi}{2}$.

But neither $\sec\theta$ nor $\tan\theta$ is defined for $\theta=\frac{3\pi}{2}$.

Hence the equation has no solution.

Example 3: Solve $\sin\theta+\cos\theta=1$.

In such equations, it is convenient to divide both the sides by the square root of the sum of the squares of the coefficients of $\sin\theta$ and $\cos\theta$. Here the coefficients are 1 in both, so divide by $\sqrt{2}$.

Thus $\frac{1}{\sqrt{2}}\sin\theta+\frac{1}{\sqrt{2}}\cos\theta=\frac{1}{\sqrt{2}}$

Hence $\cos\frac{\pi}{4}\sin\theta+\sin\frac{\pi}{4}\cos\theta=\frac{1}{\sqrt{2}}$.

$\therefore \sin\left(\theta+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}=\sin\frac{\pi}{4}$ or $\sin\left(\frac{\pi}{2}+\frac{\pi}{4}\right)$

$\therefore \theta+\frac{\pi}{4}=\frac{\pi}{4}$ or $\frac{\pi}{2}+\frac{\pi}{4}$

$\therefore \theta=0$, or $\theta=\frac{\pi}{2}$.

After checking, we see that they do satisfy the given equation. Thus the solutions are $\theta=0$, $\theta=\frac{\pi}{2}$.

Example 4: Solve $2\sec\theta=\tan\theta+\cot\theta$.

We have $\frac{2}{\cos\theta}=\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}=\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}=\frac{1}{\sin\theta\cos\theta}$

$\therefore 2\cos\theta\sin\theta-\cos\theta=0$ or $(2\sin\theta-1)\cos\theta=0$

$\therefore \sin\theta=\frac{1}{2}$ or $\cos\theta=0$

From $\sin\theta=\frac{1}{2}$, we have $\theta=\frac{\pi}{6}$, $\pi-\frac{\pi}{6}=\frac{5\pi}{6}$ and $\cos\theta=0$ gives $\theta=\frac{\pi}{2}$ or $\frac{3\pi}{2}$; but for the latter values, $\sec\theta$ and $\tan\theta$ are not defined. $\therefore$ the only solutions are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

Exercise 1

Solve the following trigonometrical equations. (Use trigonometric tables if necessary).

1. $\sin\theta\cos\theta=0$.

2. $2\sin^2\theta=1$.

3. $4\sin^2\theta-3=0$.

4. $2\cos^2\theta-\cos\theta=0$.

5. $\cos\theta\sin\theta+\sin\theta=0$.

6. $\sin\theta=\cos\theta$.

7. $4\sin\theta=\cos\theta$.

8. $4\cos\theta=\sin\theta$.

9. $2\sin^2\theta-\cos\theta-2=0$.

10. $\sin\theta=2(1-\sin\theta)$.

11. $\tan^2\theta+\tan\theta=0$.

12. $\tan^2\theta-3\tan\theta+2=0$.

13. $\tan^2\theta+\sec\theta=1$.

14. $\tan^2\theta+2\sec\theta=2$.

15. $2\sin\theta\cos\theta=\cot\theta$.

16. $2\sin x-\operatorname{cosec} x=1$.

17. $\tan x+2\cot x=3$.

18. $\cos x-\sqrt{3}\sin x=1$.

19. $\sin\theta-\cos\theta+1=0$.

20. $2\cos\theta+\sec\theta=3$.

2. Graphs of trigonometric functions.

We have seen that trigonometric functions like sin x, cos x and tan x are defined for every angle x. The only exceptions are $\frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ for tan θ. It is usual to measure x in degrees, minutes and seconds, but for mathematical purposes, it is more advantageous to measure angles in *radians*. For any real number x, there are trigonometric functions like sin x where x is an angle measured in radians. This means that for every value of x, the value of sin x is unique; but as we have seen in the last section, the converse is not true. If $\sin x=\frac{1}{2}$ the value of x in radians may be $x=\frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$ and so on. Actually the values of the trigonometric functions repeat after 2π or 360°. Thus $\sin(x+2\pi)=\sin x$, $\cos(x+2\pi)=\cos x$ and so on. That is why the trigonometric functions are called *periodic*, the period being 2π.

More generally, a function $f(x)$ is called *periodic* with period, a, if $f(x+a)=f(x)$.

Definition : If $f(x)$ has the period $a>0$ and no smaller positive period, than a is called a *primitive period* of $f(x)$.

Since $\tan(x+\pi)=\tan x$,

π is a primitive period of tan x. Similarly 2π is the primitive period of sin x and cos x. **Graphs of sin x, cos x, tan x.**

(i) $y=\sin x$

Since sin x is a periodic function with period 2π, we need draw the graph from $x=0$ to $x=2\pi$ only. Outside this range the curve will repeat itself. To draw the graph we must plot certain points which lie on the curve $y=\sin x$. First we observe that since $-1\leqslant\sin x\leqslant1$ for all x, the entire curve will lie between the lines $y=1$ and $y=-1$. We tabulate a few values.

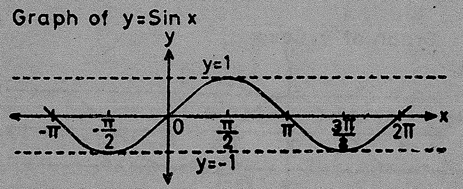

Fig. 12.1

x in radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{3\pi}{2}$	2π
$y=\sin x$	0	$\frac{1}{2}$	0·7	1	·86	0	-1	0

Plotting the above points, we get the graph as shown above. At $x=\frac{\pi}{2}$, the curve touches the line $y=1$ and at $x=\frac{3\pi}{2}$ it touches the line $y=-1$. At $x=0$, π, 2π it crosses the axis of x.

(ii) $y=\cos x$. Here again $-1\leqslant\cos x\leqslant1$. The table of coordinates for the graph of $y=\cos x$ is given below.

x in radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{3\pi}{2}$	2π
$y=\cos x$	1	·86	·7	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	0	1

Plotting the points, we get the graph as under:

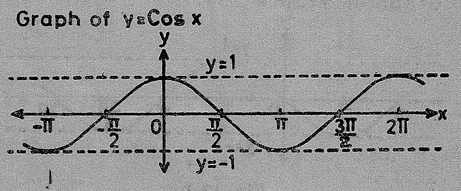

Fig. 12.2

Since $\sin\left(\frac{\pi}{2}+x\right)=\cos x$, the graph of $y=\cos x$ can be obtained by shifting the graph of $y=\sin x$, along the x-axis by a distance of $\frac{\pi}{2}$ to the left.

(iii) $y=\tan x$

Here the table of coordinates is as under:

x	0	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
y	0	1	1·73	$\propto$	-1·73	0

Since $\tan x$ has a period of π, the graph of $y=\tan x$ will repeat itself outside the range 0 to π. The graph is as under.

One of the most interesting features of these graphs is the behaviour of $\tan x$ near $\pm\frac{\pi}{2}$. When x approaches $+\frac{\pi}{2}$ but

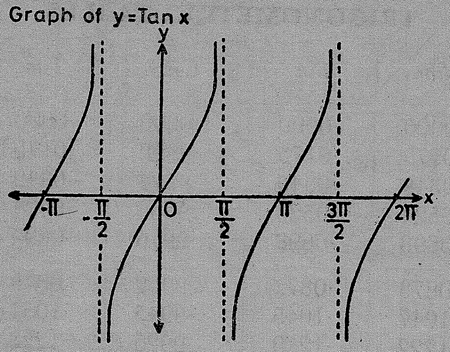

Fig. 12.3

remains always less than $+\frac{\pi}{2}$, y approaches ∞, and when x approaches $-\frac{\pi}{2}$, but all the time greater than $-\frac{\pi}{2}$ y approaches $-\infty$. These facts can be checked from the figure above.

Exercise

1. Draw the graph of $\sin 2x$ between $-\pi\leqslant x\leqslant\pi$. What is the primitive period of the function? What is the maximum value of $\sin 2x$? What is the minimum value?

2. Draw the graph of $\cos 2x$ between $-\pi\leqslant x\leqslant\pi$. What is the primitive period? What are the maximum and minimum values of the function?

3. Draw the graph of $\sin x+\cos x$.

$\left[\text{Note that } \sin\left(x+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}(\sin x+\cos x) \text{ so that if } y=\sin x+\cos x,\ y=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\right].$

4. Draw the graph of $y=2\sin x$. What are the maximum and minimum values of y?

5. Draw the graph of $y=\tan 2x$. What is the primitive period of y? What are the values of x which make y tend to infinity?

6. Prove that if $f(x)$ and $F(x)$ have the period a and the same domain, then $f(x)\pm F(x)$ has the period a.

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TRIGONOMETRIC TABLES

Degrees	*Radians*	*Sin*	*Cos*	*Tan*	*Cot*
0	·0000	·0000	1·0000	·0000	not defined
1	·0175	·0175	·9998	·0175	57·29
2	·0349	·0349	·9994	·0349	28·64
3	·0524	·0523	·9986	·0524	19·08
4	·0698	·0698	·9976	·0699	14·30
5	·0873	·0872	·9962	·0875	11·430
6	·1047	·1045	·9945	·1051	9·5144
7	·1222	·1219	·9925	·1228	8·1443
8	·1396	·1392	·9903	·1405	7·1154
9	·1571	·1564	·9877	·1584	6·3138
10	·1745	·1736	·9848	·1763	5·6713
11	·1920	·1908	·9816	·1944	5·1446
12	·2094	·2079	·9781	·2126	4·7046
13	·2269	·2250	·9744	·2309	4·3315
14	·2433	·2419	·9703	·2493	4·0108
15	·2618	·2588	·9659	·2679	3·7321
16	·2793	·2756	·9613	·2867	3·4874
17	·2967	·2924	·9563	·3057	3·2709
18	·3142	·3090	·9511	·3249	3·0777
19	·3316	·3256	·9455	·3443	2·9042
20	·3491	·3420	·9397	·3640	2·7475
21	·3665	·3584	·9336	·3839	2·6051
22	·3840	·3746	·9272	·4040	2·4751
23	·4014	·3907	·9205	·4245	2·3559
24	·4189	·4067	·9135	·4452	2·2460
25	·4363	·4226	·9063	·4663	2·1445
26	·4538	·4384	·8988	·4877	2·0503
27	·4712	·4540	·8910	·5095	1·9626
28	·4887	·4695	·8829	·5317	1·8807
29	·5061	·4848	·8746	·5543	1·8040

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Degrees	*Radians*	*Sin*	*Cos*	*Tan*	*Cot*
30	·5236	·5000	·8660	·5774	1·7321
31	·5411	·5150	·8572	·6009	1·6643
32	·5585	·5299	·8480	·6249	1·6003
33	·5760	·5446	·8387	·6494	1·5399
34	·5934	·5592	·8290	·6745	1·4826
35	·6109	·5736	·8192	·7002	1·4281
36	·6283	·5878	·8090	·7265	1·3764
37	·6458	·6018	·7986	·7536	1·3270
38	·6632	·6157	·7880	·7813	1·2799
39	·6807	·6293	·7771	·8098	1·2349
40	·6981	·6428	·7660	·8391	1·1918
41	·7156	·6561	·7547	·8693	1·1504
42	·7330	·6691	·7431	·9004	1·1106
43	·7505	·6820	·7314	·9325	1·0724
44	·7679	·6947	·7193	·9657	1·0355
45	·7854	·7071	·7071	1·0000	1·0000
46	·8029	·7193	·6947	1·0355	·9657
47	·8203	·7314	·6820	1·0724	·9325
48	·8378	·7431	·6691	1·1106	·9004
49	·8552	·7547	·6561	1·1504	·8693
50	·8727	·7660	·6428	1·1918	·8391
51	·8901	·7771	·6293	1·2349	·8098
52	·9076	·7880	·6157	1·2799	·7813
53	·9250	·7986	·6018	1·3270	·7536
54	·9425	·8090	·5878	1·3764	·7265
55	·9599	·8192	·5736	1·4281	·7002
56	·9774	·8290	·5592	1·4826	·6745
57	·9948	·8387	·5446	1·5399	·6494
58	1·0123	·8480	·5299	1·6003	·6249
59	1·0297	·8572	·5150	1·6643	·6009

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Degrees	Radians	Sin	Cos	Tan	Cot
60	1·0472	·8660	·5000	1·7321	·5774
61	1·0647	·8746	·4848	1·8040	·5543
62	1·0821	·8829	·4695	1·8807	·5317
63	1·0996	·8910	·4540	1·9626	·5095
64	1·1170	·8988	·4384	2·0503	·4877
65	1·1345	·9063	·4226	2·1445	·4663
66	1·1519	·9135	·4067	2·2460	·4452
67	1·1694	·9205	·3907	2·3559	·4245
68	1·1868	·9272	·3746	2·4751	·4040
69	1·2043	·9336	·3584	2·6051	·3839
70	1·2217	·9397	·3420	2·7475	·3640
71	1·2392	·9455	·3256	2·9042	·3443
72	1·2566	·9511	·3090	3·0777	·3249
73	1·2741	·9563	·2924	3·2709	·3057
74	1·2915	·9613	·2756	3·4874	·2867
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81	1·4137	·9877	·1564	6·3138	·1584
82	1·4312	·9903	·1392	7·1154	·1405
83	1·4486	·9925	·1219	8·1443	·1228
84	1·4661	·9445 9945	·1045	9·5144	·1051
85	1·4835	·9962	·0872	11·43	·0875
86	1·5010	·9976	·0698	14·30	·0699
87	1·5184	·9986	·0523	19·08	·0524
88	1·5359	·9994	·0349	28·64	·0349
89	1·5533	·9998	·0175	57·29	·0175
90	1·5708	1·0000	·0000	not defined	·0000

ANSWERS

CHAPTER 5

Exercise 1

1. (i) $\frac{3\pi}{2}$, (ii) $\frac{7\pi}{4}$, (iii) $-\frac{14\pi}{3}$, (iv) $\frac{\pi}{360}$.
(v) $-\frac{81\pi}{4}$, (vi) $\frac{\pi}{648000}$.

3. (i) 15°, (ii) 252°, (iii) −1530°, (iv) 114°, (v) 286°, (vi) 25° 42′ 51″.

4. (i) $\frac{\pi}{3}$, (ii) $\frac{3\pi}{5}$, (iii) $\frac{5\pi}{7}$.

5. 40 m. **6.** 21 cm. (approximately).

7. (i) $\frac{1}{200}$ second; $\frac{1}{20\pi}$ sec.

8. 1·886 km. **9.** $\frac{5}{108}$ radian. **10.** 2·5 radians.

11. 113 km. **12.** $\frac{7\pi}{8}$ or 1·2 km. **13.** 5·40 km. per hour.

Exercise 2

1. (1) $\sin\theta=\frac{3}{\sqrt{13}}$, $\cos\theta=-\frac{2}{\sqrt{13}}$, $\tan\theta=-\frac{3}{2}$
(2) $\sin\theta=\frac{4}{\sqrt{41}}$, $\cos\theta=\frac{5}{\sqrt{41}}$, $\tan\theta=\frac{4}{5}$.
(3) $\sin\theta=-\frac{4}{5}$, $\cos\theta=\frac{3}{5}$, $\tan\theta=-\frac{4}{3}$.
(4) $\sin\theta=-\frac{4}{5}$, $\cos\theta=-\frac{3}{5}$, $\tan\theta=\frac{4}{3}$.
(5) $\sin\theta=-\frac{24}{25}$, $\cos\theta=\frac{7}{25}$, $\tan\theta=-\frac{24}{7}$.
(6) $\sin\theta=\frac{12}{13}$, $\cos\theta=-\frac{5}{13}$, $\tan\theta=-\frac{12}{5}$.

2. (i) $\cos^2\theta$, (ii) $\sec^2\theta$, (iii) 1, (iv) $\sin\theta$, (v) $\cos\theta$, (vi) $\sin\theta$, (vii) $\cot\theta$.

3. tan $\theta=-\frac{3}{4}$, cot $\theta=-\frac{4}{3}$, cosec $\theta=\frac{5}{3}$, sec $\theta=-\frac{5}{4}$.

4. cos $\theta=\pm\dfrac{\sqrt{21}}{5}$, tan $\theta=\pm\dfrac{2}{\sqrt{21}}$, cot $\theta=\pm\dfrac{\sqrt{21}}{2}$.

5. 2nd or 3rd, $-\frac{3}{8}$. 6. first or third, $-\frac{2}{3}$.

7. sin θ is negative, cos θ negative and tan θ positive.

8. sin θ is negative, cos θ positive and tan θ negative.

Exercise 3

1. (a) $\theta=30°$, (b) $\theta=45°$, (c) $\theta=45°$.

2. (i) $\theta=\dfrac{2\pi}{3}$, (ii) $\theta=\dfrac{3\pi}{4}$, (iii) $\theta=\pi$.

3. (i) $\theta=\dfrac{3\pi}{2}$, (ii) $\theta=\dfrac{4\pi}{3}$, (iii) $\theta=\dfrac{4\pi}{3}$.

4. (i) $\theta=\dfrac{7\pi}{4}$, (ii) $\theta=\dfrac{7\pi}{4}$, (iii) $\theta=2\pi$.

5. $\frac{\sqrt{3}}{2}$. 6. $\frac{1}{4}$. 7. $\frac{1}{2}$. 8. 2. 9. $-\frac{2}{3}$.
10. $\frac{1}{2}$. 14. 5. 15. 3.

CHAPTER 6

Exercise 1

1. 5. 2. 13. 3. 25. 4. $\sqrt{34}$. 5. $\frac{1}{2}\sqrt{a^2+b^2}$.

6. $\sqrt{2(a^2+b^2)}$. 7. $\dfrac{\sqrt{a^2+b^2}}{ab}$. 8. ± 8.

9. 5. 10. 3. 11. 3. 12. 8. 13. $2b$.
14. 5. 15. 10. 16. $5\sqrt{2}$. 17. 5. 18. 15.
20. 18. 26. $5\frac{1}{4}$. 27. (1, 0). 29. (1, 0).

Exercise 2

1. (0, 0). 2. $(2\frac{1}{3}, 1)$. 3. $(3, 1\frac{2}{3})$. 4. $(3\frac{4}{5}, -\frac{6}{5})$.
5. $(2\frac{1}{2}, 4\frac{1}{4})$. 6. $(1\frac{1}{5}, -\frac{1}{5})$. 7. $(5\frac{1}{2}, -4\frac{1}{2})$. 8. (−5, −8)
9. $(-2\frac{1}{3}, -3\frac{1}{3})$. 10. $(-2, 5\frac{2}{3})$. 11. (1, −11). 12. $(2\frac{1}{2}, -3\frac{3}{4})$.

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13. $(a, b), (c, d)$; 1 : 2. 14. $(a, b), (c, d)$; −2 : 1.
15. (4, 3), (3, 2); $m : -n$. 16. $(a, b), (c, d)$; 1 : 1.
17. $(5, 1\frac{2}{3}), (7, 4\frac{1}{3})$. 18. $(-2\frac{1}{2}, -5), (-1, -2), (\frac{1}{2}, 1)$
19. $x=-14, y=6$.

CHAPTER 7

Exercise 1

1. 3, 2, $\frac{3}{2}$. 2. 2, −3, $+\frac{3}{2}$. 3. 5, 0, 0.
4. 0, 7, not defined. 5. 3, 4, $\frac{4}{3}$. 6. 0. 7. 0.
8. not defined. 9. 1. 10. $-\frac{1}{2}$. 11. not defined.
3. 2, 2 collinear. 14. $-\frac{2}{5}, \frac{5}{2}$; $mm'=-1$.
3. $-\frac{2}{3}$. 19. $-\frac{1}{2}$, 2. 21. $y-x=1$.
2. $3y-x+8=0$. 23. $2y-x=5$.
4. $2y+3x=13$. 25. $4x-3y=11$.
6. (i) $y+x=5$, (ii) $y-3x+4=0$, (iii) $7y+2x=3$, (iv) $x-y=1$,
(v) $bx+ay=ab$.
7. $y=3x$. 28. $y+x=0$. 29. $2y-x=3$, $y+2x=4$.
30. $4y-x+6=0$.
$\dfrac{x}{2}+\dfrac{y}{3}=1$; $2x-3y=0$. 32. $\dfrac{y}{5}-\dfrac{x}{3}=1$.
3. $3x+5y=8$; $x+4y=5$. 34. $5x+2y=16$.
$28x-21y+12=0$.

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ercise 2

$x\cos\alpha+y\sin\alpha=p$ where

(i) $\cos\alpha=\dfrac{1}{\sqrt{5}}$, $\sin\alpha=-\dfrac{2}{\sqrt{5}}$ and $p=0$

(ii) $\cos\alpha=1$, $\sin\alpha=0$ and $p=-\frac{4}{3}$

(iii) $\cos\alpha=\frac{3}{5}$, $\sin\alpha=\frac{4}{5}$ and $p=1$

(iv) $\cos\alpha=\frac{3}{5}$, $\sin\alpha=-\frac{4}{5}$ and $p=\frac{7}{5}$

(v) $\cos\alpha=-\dfrac{2}{\sqrt{13}}$, $\sin\alpha=\dfrac{3}{\sqrt{13}}$ and $p=\dfrac{5}{\sqrt{13}}$

(vi) $\alpha=\frac{2\pi}{3}$ and $p=3$

(vii) $\cos\alpha=\frac{-1}{\sqrt{2}}$, $\sin\alpha=\frac{1}{\sqrt{2}}$ and $p=5$

(viii) $\cos\alpha=\frac{5}{13}$, $\sin\alpha=\frac{12}{13}$ and $p=1$.

2. (i) $\frac{11}{\sqrt{13}}$, (ii) $\frac{7}{\sqrt{34}}$, (iii) 2.

3. (i) $1\frac{3}{5}$, (ii) $\frac{9}{\sqrt{13}}$, (iii) $\frac{14}{\sqrt{5}}$ (iv) $\frac{14}{\sqrt{10}}$ (v) 6,

4. (i) $\frac{10}{13}$, (ii) $5\frac{10}{13}$, (iii) $2\frac{8}{13}$, (iv) $1\frac{1}{13}$, (v) $\frac{7}{13}$,
(i), (ii), (iv) lie on the same side as the origin.

5. $\frac{17\sqrt{5}}{5}$. **6.** $\frac{5\sqrt{2}}{2}$. **7.** $\frac{38}{\sqrt{29}}$ **8.** 2·2. **9.** $\frac{3\sqrt{2}}{2}$.

10. On the same side. **12.** 4

13. $x+3y=15$; $x+3y+5=0$.

14. $x(\sqrt{2}-1)-y(2\sqrt{2}-3)+3\sqrt{2}+5=0$;
$x(\sqrt{2}+1)-y(2\sqrt{2}+3)+3\sqrt{2}-5=0$.

15. $x(2\sqrt{2}-3)-y(\sqrt{2}+1)+5\sqrt{2}-7=0$;
$x(2\sqrt{2}+3)-y(\sqrt{2}-1)+5\sqrt{2}+7=0$.

16. $x+y=7$; $x-y=1$. **17.** $2x+3y=120$; $3x-2y+28=0$.

18. $3x-11y+78=0$; $11x+3y+26=0$.

19. $\frac{13}{\sqrt{5}}$, $\frac{26}{\sqrt{73}}$, $\frac{26}{\sqrt{37}}$.

20. $x=2$, $9y-3x+2=0$, $9y+3x=10$.

Exercise 3

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1. $19x+38y+17=0$.
2. $25x-2y+4=0$.
3. $100x+59y+300=0$.
4. $4x-19y+31=0$.
5. $2x+3y+9=0$.
6. $9x+77y=95$.
7. $34x+119y=203$.
8. $50x-75y+17=0$.
9. $27x+18y+17=0$.
10. $19y+51=0$.
11. $x=1$.
12. $x+1=0$.

Exercise 4

1. One is ⊥ the other. yes. $\sqrt{29}$. **2.** (i) $4y-2x=9$,
(ii) $16x+6y=7$, (iii) $x+1=0$, (iv) $8y-6x+25=0$.

3. $y=x$. **4.** $x^2+y^2=25$.

5. $(x-1)^2+(y-2)^2=4$; (3, 2) and (1, 0) lie on it.

6. $x^2-2x-4y+5=0$ yes.

7. $3(x^2+y^2)+2x+2y-2=0$. **8.** $3x^2-y^2+10x-25=0$.

9. $(x^2+y^2)-5x+4=0$. **10.** $3(x^2+y^2)+6x+8y-25=0$.

11. $x^2+y^2=a^2$. **13.** $2(y^2-x^2)=1$.

14. $x=0$. **15.** $x^2+y^2-2x+6y+5=0$.
$x^2+y^2-4x+2y=0$.

16. $x^2+y^2=4$; The condition is satisfied as a limiting case.

17. $x-y+2=\pm2\sqrt{2}$. **18.** $x-2y+8=\pm\sqrt{5}$.

Exercise 5

1. 7·5 sq. units. **2.** 0. **3.** 6·5 sq. units.
4. 5 sq. units. **5.** 16 sq. units. **6.** 9 sq. units.
7. $\frac{1}{2}(a^2+b^2+c^2-bc-ca-ab)$ sq. units.
8. 1 sq. unit. **9.** 6 sq. units. **10.** 6 sq. units.
11. $\frac{22}{\sqrt{53}}$; 11; $\sqrt{53}$ **12.** $y=3$. or $y=-1$ **14.** $x+y=5$.
15. $3x-y-9=\pm6$.

CHAPTER 8

Exercise 1

1. $x^2+y^2-2x-2y-7=0$.
2. $x^2+y^2-4x-6y+12=0$.
3. $x^2+y^2-4x+4y-1=0$.
4. $x^2+y^2-2x+6y+6=0$.
5. $x^2+y^2+4x+6y-12=0$.
6. $x^2+y^2-8x+2y=0$.
7. $x^2+y^2-4x+10y=0$.
8. $x^2+y^2-3x-y=0$.
9. $x^2+y^2+2x-6y-3=0$.
10. $x^2+y^2-6x-4y-28=0$.
11. (2, 0); 2. **12.** (0, 4); 4. **13.** (2, −1); $\sqrt{5}$.
14. (−2, −2); $2\sqrt{2}$. **15.** (−2, 0); 1. **16.** (0, −3); 4.

17. $(-1, -2)$; 3. **18.** $(-3, 2)$; 5. **19.** $(6, -2)$; 5.
20. $(-1\frac{1}{2}, -2\frac{1}{2})$; 3. **21.** $(-\frac{1}{2}, -1)$; 1. **22.** not real.
23. $5y+x=2$. **24.** $x^2+y^2-x+y-2=0$.
25. $3x^2+3y^2+2x-16y+5=0$.
26. $2(x^2+y^2)+4x-13y+5=0$.
27. $x^2+y^2-2x-2y-2=0$.
28. $x^2+y^2-3x-2y=0$.
29. $x^2+y^2-4x-6y+8=0$.
30. $(10, 24)$; $(10, 0)$ and $(0, 24)$.
31. $x^2+y^2-8x-8y+7=0$.

Exercise 2

1. $-3x+4y=25$. **2.** $x\sqrt{3}+y=3$. **3.** $2x+y=5$.
4. $2x+\sqrt{5}y=9$. **5.** $2\sqrt{2}x+\sqrt{3}y=11$. **6.** $x-4y=17$.
7. $5\sqrt{3}x-5y=8$. **8.** $y=5$. **9.** $x-y+1=0$.
10. $3x-4y=24$; $3x+4y=8$. **11.** $2x+y=10$; $2x-y=10$.
12. $x+y=6$; $x+y=-6$. **13.** $\sqrt{5}$. **14.** $\sqrt{21}$.
15. $x_1^2+y_1^2-9$; $x^2+y^2=25$. **16.** $x^2+y^2+2x-4y-4=0$.
17. $2x^2+2y^2+3x-2y-17=0$.
18. $(2, -2)$ inside; $(1, 1)$ on it; $(3, 2)$ outside.
19. $y+2=0$. **21.** $-3x+4y=25$; $x=5$; $x=-5$.
22. $10(x-4)^2+10(y+2)^2=196$. **23.** $(x-3)^2+(y-5)^2=4$.
25. $(x-19)^2+(y+7)^2=325$. **26.** (a) $6x+8y+21=0$.
27. $(-1, -3)$. **28.** $(3, 2)$. (b) $4x+3y-9=0$.

CHAPTER 9

Exercise 1

1. $-\cot\theta$, $\cot\theta$, $-\cos\theta$, $\sin\theta$.
2. $-\tan\theta$, $-\cot\theta$, $-\sin\theta$, $-\sec\theta$.
3. $-\sin\theta$, $\cos\theta$, $-\tan\theta$, $-\cot\theta$.
4. $-\sin\theta$, $-\cos\theta$, $\tan\theta$, $\cot\theta$.
5. $-\cos\theta$, $-\sin\theta$, $\cot\theta$, $\tan\theta$.

6. $\dfrac{\sqrt{3}}{2}$, $-\frac{1}{2}$, $-\dfrac{\sqrt{3}}{2}$, $-\dfrac{\sqrt{3}}{2}$, $-\frac{1}{2}$.
7. $\dfrac{1}{\sqrt{2}}$, $-\dfrac{1}{\sqrt{2}}$, $-\dfrac{1}{\sqrt{2}}$, $-\dfrac{1}{\sqrt{2}}$.
8. 0, $-\dfrac{1}{\sqrt{2}}$, $-\dfrac{1}{\sqrt{2}}$, $-\dfrac{1}{\sqrt{2}}$.
9. $\dfrac{\sqrt{3}}{2}$, $-\frac{1}{2}$, $-\dfrac{\sqrt{3}}{2}$, $-\frac{1}{2}$.
10. $-1, 1, -1, 1$.
11. 3·73, $-\dfrac{1}{3{\cdot}73}$, 3·73.
12. $2-\sqrt{3}$.

CHAPTER 10

Exercise 1

1. 76·425 m. **2.** 30°. **3.** height 6·928 m.; dist. 4 m.
4. 37·3 m. **5.** 17·32 m. **6.** 21·0 m. (approx.)
7. 18·30 m. **8.** 31·4 cm. **9.** 25·548 m.
10. 42·26 m. **11.** First 31·177 m.; second 46·767 m.
12. 46·65 m. **13.** 141 m. **14.** 14·142 m.p.h.
15. 51° (approx.) with N.W. **16.** 14·3 m.; 11·9 m.
17. 43·3 m.; 25 m. from one. **18.** 94·65 m.; 54·65 m.
20. 5177·6 km. **21.** 4·3 km. (approximately).

Exercise 2

1. (i) $a=5{\cdot}8$ cm, $b=3{\cdot}8$ cm (ii) $a=5{\cdot}9$ cm, $c=5{\cdot}6$ cm,
(iii) $b=3{\cdot}4$ cm; $c=6{\cdot}6$ cm.
2. $c=28°$, $b=5{\cdot}5$ cm, $c=11$ cm.
3. $C=62°$, $b=2{\cdot}7$ cm, $c=3{\cdot}8$ cm.
4. $B=75°$, $a=7{\cdot}7$ cm, $c=5{\cdot}8$ cm.
5. $C=80°$, $a=9{\cdot}7$ cm, $b=5{\cdot}1$ cm.
6. $B=80°$, $a=5{\cdot}8$ cm, $c=5$ cm.

7. $A=20°$, $b=6·5$ cm, $c=11·8$ cm.
8. $B=40°$, $a=b=3·1$ cm.
9. $C=85°$, $a=6·5$ cm, $c=11·4$ cm.
10. 7·4 cm. **11.** 7·1 cm; 9·9 cm. **12.** 9·2 km; 7·7 km.
13. 21·8 cm. **14.** $A=56°$, $B=41°$, $C=83°$.
15. 8·2 cm, $B=76°$, $C=34°$.
16. $b=5·35$ cm, $\angle A=46°$, $\angle C=62°$.
17. $A=61°$, $B=53°$, $C=66°$.
18. 16·6 cm, 21·5 cm, 43°, 32°.
19. 3·366 km.
20. 102°, 78°.

CHAPTER 12

Exercise 1

1. $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. **2.** $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.
3. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$. **4.** $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{3}, \frac{5\pi}{3}$.
5. $0, \pi$. **6.** $\frac{\pi}{4}, \frac{5\pi}{4}$. **7.** 14°, 194°.
8. 76°, 256°. **9.** $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}$.
10. 42°, 138°. **11.** $0, \pi, \frac{3\pi}{4}, \frac{7\pi}{4}$.
12. $\frac{\pi}{4}, \frac{5\pi}{4}$, 64°, 244°. **13.** $0, \frac{2\pi}{3}, \frac{4\pi}{3}$.
14. 0, 109°, 251°. **15.** $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.
16. $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$. **17.** $\frac{\pi}{4}, \frac{5\pi}{4}$, 64°, 244°.
18. $0, \frac{4\pi}{3}$. **19.** $0, \frac{3\pi}{2}$. **20.** $0, \frac{\pi}{3}, \frac{5\pi}{3}$.